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THE MATHEMATICAL ASSOCIATION

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★

'I hold every man a debtor to his profession, from the which as men of course do seek to receive countenance and profit, so ought they of duty to endeavour themselves by way of amends to be a help and an ornament thereunto.'
BACON

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TRIGONOMETRY RETOLD

(AN EXPOSITORY ARTICLE)

BY S. PARAMESWARAN

Trigonometry, like most abstract sciences, has both an operational side and a quantitative side. The customary discussion is mainly quantitative, based as it is on the concept of the radian and the definition of sine and cosine as ratios of the sides of a right-angled triangle. The operational side has been touched upon by V. Naylor in his "Note on the Nature of Trigonometry" (1). Starting with the hypothesis that a function $\lambda(x)$ exists which has the property

$$\lambda(x) \,.\, \lambda(y) = \lambda(x+y)\,; \quad \lambda(0) = 1, \qquad \text{.......................(A)}$$

and introducing two functions $f(x)$ and $\phi(x)$ defined by the relation

$$\lambda(x) = f(x) + i\phi(x),$$

Naylor has derived certain results which can be looked upon as generalised trigonometrical formulae, for if we specialise f and ϕ as cosine and sine, the results established turn out to be the standard results in circular functions.

The hypothesis (A) is equivalent to

$$\left.\begin{aligned} f(x+y) &= f(x) \,.\, f(\ \ ) - \phi(x) \,.\, \phi(y), \\ \phi(x+y) &= \phi(x) f(y) + f(x) \,.\, \phi(y), \\ f(0) &= 1, \quad \phi(0) = 0. \end{aligned}\right\} \qquad \text{.....................(B)}$$

The purpose of this note is to show that, with the assumption of a single equation (C) in f and ϕ, all the results in the aforesaid "Note" can be derived. Further, assuming f and ϕ to be differentiable functions their values can be determined.

SECTION I

Let f and ϕ be non-constant real functions such that, always

$$f(x-y) = f(x) \,.\, f(y) + \phi(x) \,.\, \phi(y). \qquad \text{........................(C)}$$

From the symmetric roles played by x and y in (C) it follows that:

f is an even function. ....(i)

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$$\lambda(x)\,.\,\lambda(y)=\lambda(x+y)\,;\quad \lambda(0)=1, \quad\ldots\ldots\ldots\ldots\ldots\ldots\text{(A)}$$

and introducing two functions $f(x)$ and $\phi(x)$ defined by the relation

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The hypothesis (A) is equivalent to

$$\left.\begin{aligned} f(x+y)&=f(x)\,.\,f(y)-\phi(x)\,.\,\phi(y),\\ \phi(x+y)&=\phi(x)f(y)+f(x)\,.\,\phi(y),\\ f(0)&=1,\quad \phi(0)=0. \end{aligned}\right\}\quad\ldots\ldots\ldots\ldots\ldots\ldots\text{(B)}$$

The purpose of this note is to show that, with the assumption of a single equation (C) in f and ϕ, all the results in the aforesaid "Note" can be derived. Further, assuming f and ϕ to be differentiable functions their values can be determined.

SECTION I

Let f and ϕ be non-constant real functions such that, always

$$f(x-y)=f(x)\,.\,f(y)+\phi(x)\,.\,\phi(y). \quad\ldots\ldots\ldots\ldots\ldots\ldots\text{(C)}$$

From the symmetric roles played by x and y in (C) it follows that:

$$f \text{ is an even function.} \quad\ldots\ldots\ldots\ldots\ldots\ldots\text{(i)}$$

The relation (C) shows that $\phi(x) \cdot \phi(y) = \phi(-x) \cdot \phi(-y)$. Hence $\phi(x)$ cannot be the sum of an even and an odd function.

$\phi(x)$ cannot be an even function also; for if it were so, (C) yields the impossible result $f(2x) = f(0)$ for all values of x, when y is replaced by $-x$ and $+x$ successively. Therefore

$$\phi(x) \text{ is an odd function.} \qquad \text{(ii)}$$

Being an odd function

$$\phi(0) = 0. \qquad \text{(iii}a\text{)}$$

Taking $y = 0$ in (C), we get $f(x) = f(x) \cdot f(0) + \phi(x) \cdot \phi(0)$, and since $f(x)$ is not constantly zero it follows that $f(0) = 0$ is impossible and so

$$f(0) = 1. \qquad \text{(iii}b\text{)}$$

Setting $y = x$, (C) reduces to the form

$$[f(x)]^2 + [\phi(x)]^2 = 1. \qquad \text{(iv)}$$

From (C), using (i) and (ii), we obtain

$$f(x+y) = f(x) \cdot f(y) - \phi(x) \cdot \phi(y). \qquad \text{(v)}$$

On replacing x by $(x+y)$ in (C) and simplifying, we get

$$\phi(x+y) = \phi(x) \cdot f(y) + f(x) \cdot \phi(y), \qquad \text{(vi)}$$

whence follows the result:

$$\phi(x-y) = \phi(x) \cdot f(y) - f(x) \cdot \phi(y). \qquad \text{(vii)}$$

The following results are easily deduced from those above:

$$\left.\begin{aligned} f(2x) &= [f(x)]^2 - [\phi(x)]^2 \\ \phi(2x) &= 2\phi(x) \cdot f(x), \\ \phi(2x) \pm \phi(2y) &= 2\phi(x \pm y) \cdot f(x \mp y), \\ f(2x) + f(2y) &= 2f(x+y) \cdot f(x-y), \\ f(2x) - f(2y) &= -2\phi(x+y) \cdot \phi(x-y). \end{aligned}\right\} \qquad \text{(viii)}$$

Defining $\psi(x)$ by the relation $\psi(x) = \dfrac{\phi(x)}{f(x)}$ we obtain

$$\psi(x+y) = \frac{\psi(x) + \psi(y)}{1 - \psi(x) \cdot \psi(y)} \quad \text{and} \quad \psi(x-y) = \frac{\psi(x) - \psi(y)}{1 + \psi(x) \cdot \psi(y)}. \qquad \text{(ix)}$$

Section II

In this section we assume that there is a value $t \neq 0$, such that $\phi(t) = 1$, and therefore $f(t) = 0$, because of (iv).

Setting $y = t$ in (C) and in (vii), we have

$$f(x-t) = \phi(x) \quad \text{and} \quad \phi(x-t) = -f(x).$$

$$\therefore \ \phi(x) = -f(x+t) = -\phi(x+2t) = f(x+3t) = \phi(x+4t) = -f(x+5t).$$

Hence $\phi(x)$ and $f(x)$ have a period $= 4t$. (x)

If we take $\lambda(x) = f(x) + i\phi(x)$, then

$$\lambda(x) \cdot \lambda(y) = \lambda(x+y)$$

follows from (v) and (vi).

This, again, leads us to De Moivre's theorem (general form).

$$[f(x) + i\phi(x)]^n = [\lambda(x)]^n = \lambda(nx) = f(nx) + i\phi(nx).$$

If, now, we specialise $f(x)$, $\phi(x)$, $\psi(x)$ as $\sin x$, $\cos x$, $\tan x$ respectively, the results proved above reduce to the standard formulae in circular functions.

SECTION III

So far, we have assumed f and ϕ to be real functions only. If, further, we assume that the functions are (once) differentiable, Abel's method quoted by J. C. W. D. la Bere in a recent note [2] gives the solution of the functional equation,

$$f(x-y)=f(x)\,.\,f(y)+\phi(x)\,.\,\phi(y)$$

in the form

$$\phi(x)=Ae^{mx}+Be^{nx},$$

$$f(x)=kAe^{mx}+lBe^{nx}.$$

If we substitute these solutions in the equation, we have

$$kAe^{m(x-y)}+lBe^{n(x-y)}$$

$$=A^2(k^2+1)e^{m(x+y)}+B^2(l^2+1)e^{n(x+y)}+(kl+1)AB\,.\,(e^{mx+ny}+e^{nx+my}),$$

whence $m=-n$;

$$A^2(k^2+1)=B^2(l^2+1)=0,$$

$$kA=lB=(kl+1)AB.$$

We can suppose A and B to be non-zero ; then

$$k=\pm i\,;\quad l=\mp i,\quad A=\pm\frac{1}{2i}\,;\quad B=\mp\frac{1}{2i}\,;$$

$$\therefore\ \phi(x)=\pm\frac{1}{2i}(e^{mx}-e^{-mx})\quad\text{and}\quad f(x)=\frac{1}{2}(e^{mx}+e^{-mx}),$$

since ϕ and f are real, $mx=iax$, a being real.

$$\therefore\ \phi(x)=\sin ax\quad\text{and}\quad f(x)=\cos ax.$$

Thus $\phi(x)=\sin ax$ and $f(x)=\cos ax$ are the only real, differentiable functions satisfying the functional equation given above.

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2. *The Mathematical Gazette*, XL (1956), Note 2598, (pp. 130–1).

GLEANINGS FAR AND NEAR

1909. 46 YEARS TO MAKE UP HIS MIND?

The power of rapid and logical analysis is—after the ability accurately to count up to 13!—the most valuable asset a bridge player can develop.—From the Bridge Column, *Daily Telegraph*, Mon. January 9, 1956. [Per Mr. E. Harding.]

1910. Bischoff, and later others, showed that the brain weight of the female in relation to the male's brain weight is as 90 to 100, whereas her body weight is to the male's only as 83 to 100. Now if we were to raise the female's body weight to the equivalent proportion of the male, namely, 100 units, then one would have to add 17 units to the existing 90 for the female proportion of the brain to that of the male, giving us a figure of 107 for the female as compared with 100 brain units for the male.—Ashley Montague, *The Natural Superiority of Woman*. [Per Mr. R. E. Brownrigg.]

TWO THEOREMS ON PARTITIONS

By Richard K. Guy

Theorem 1. *$p_1(n) = p_2(n)$, where $p_1(n)$ is the number of partitions of n into odd parts greater than unity, and $p_2(n)$ is the number of partitions of n into unequal parts of which the greatest differ by unity.*

Theorem 2. *$p_1(n) = p_3(n)$, where $p_3(n)$ is the number of partitions of n into unequal parts which are not powers of two.*

The first theorem appeared as Problem 228 in *Mathematics Magazine*, 28 (1954–5), 3 (Jan.–Feb., 1955), 160, and as no proof was forthcoming, was presumably discovered empirically by the proposer, Howard D. Grossman. The second theorem emerged from the second proof of Theorem 1. All results can be found in, or follow immediately from, Chapter XIX of Hardy and Wright's *An Introduction to the Theory of Numbers*, 3rd ed., Oxford, 1954. We first establish a

Lemma. *$p_u(n) - p_u(n-1) = p_2(n)$, where $p_u(n)$ is the number of partitions of n into unequal parts.*

Proof: In the graphical representation of a partition into unequal parts, the first two rows differ either by more than one, as in (*a*), or by exactly one, as in (*b*). In the former case we can remove the top right-hand corner, and

$$(a)\quad \begin{array}{l} \text{x x x x x x x x x x} \\ \text{x x x x x x x} \\ \text{x x x x} \\ \text{x} \end{array} \qquad\qquad (b)\quad \begin{array}{l} \text{x x x x x x x x} \\ \text{x x x x x x x} \\ \text{x x x x} \\ \text{x x} \\ \text{x} \end{array}$$

leave a partition of $n-1$ into unequal parts. In the latter case we cannot. This establishes a (1, 1) correspondence between the partitions $p_u(n)$ and the partitions $p_u(n-1)$ *together with* the partitions $p_2(n)$.

First proof of Theorem 1. The generating function for $p_u(n)$, *i.e.* $\sum_{n=0}^{\infty} p_u(n) \,.\, x^n$, where $p_u(0) = 1$, is $(1+x)(1+x^2)(1+x^3) \ldots$. The generating function for $p_u(n) - p_u(n-1)$, and hence, by the lemma, for $p_2(n)$, is thus

$$\begin{aligned} &(1-x)(1+x)(1+x^2)(1+x^3)\ldots \\ &= (1-x)\,\frac{1-x^2}{1-x}\cdot\frac{1-x^4}{1-x^2}\cdot\frac{1-x^6}{1-x^3}\cdot\frac{1-x^8}{1-x^4}\ldots \\ &= \frac{1}{(1-x^3)(1-x^5)(1-x^7)\ldots} \end{aligned}$$

which is the generating function for $p_1(n)$.

Second proof of Theorem 1. Any positive integer m can be expressed uniquely in the binary scale as

$$m = 2^a + 2^b + 2^c + \ldots \quad (0 \leqslant a < b < c < \ldots).$$

Hence a partition of n into odd parts can be written as

$$\begin{aligned} n &= m_1 \,.\, 1 + m_3 \,.\, 3 + m_5 \,.\, 5 + \ldots \\ &= (2^{a_1} + 2^{b_1} + \ldots) \,.\, 1 + (2^{a_3} + 2^{b_3} + \ldots) \,.\, 3 + \\ &\qquad (2^{a_5} + 2^{b_5} + \ldots) \,.\, 5 + \ldots, \end{aligned}$$

and there is a (1, 1) correspondence between this and the partition of n into the unequal parts

$$2^{a_1}, 2^{b_1}, \dots, 2^{a_2} . 3, 2^{b_2} . 3, \dots, 2^{a_3} . 5, 2^{b_3} . 5, \dots, \dots,$$

so that $p_u(n) = p_s(n)$, the number of partitions of n into odd parts. Expressed in terms of generating functions, this result is

$$(1+x)(1+x^2)(1+x^3)\dots = \frac{1}{(1-x)(1-x^3)(1-x^5)\dots}.$$

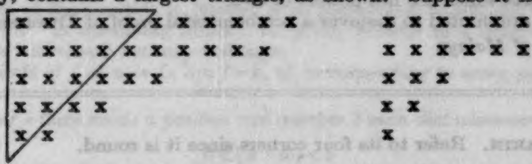
Multiply both sides by $1-x$, and use the lemma.

Third proof of Theorem 1: This is combinatorial, and includes a proof of a theorem of Euler, viz.

$$(1+x)(1+x^2)(1+x^3)\dots = 1 + \frac{x}{1-x} + \frac{x^3}{(1-x)(1-x^2)} + \frac{x^6}{(1-x)(1-x^2)(1-x^3)} + \dots$$

the indices in the numerators being triangular numbers.

The left-hand side enumerates $p_u(n)$. Any such partition, when represented graphically, contains a largest triangle, as shown. Suppose it has side m,



and thus contains $\frac{1}{2}m(m+1)$ crosses. The remainder (shown separately on right) form a partition of $n - \frac{1}{2}m(m+1)$ into at most m parts, or, reading by columns instead of rows, into parts not exceeding m. These are enumerated by

$$\frac{x^{\frac{1}{2}m(m+1)}}{(1-x)(1-x^2)\dots(1-x^m)}.$$

Summing over m proves Euler's theorem.

If the two greatest parts of this partition differ by unity, the remainder (after removal of the triangle) form a partition of $n - \frac{1}{2}m(m+1)$ into (reading by columns) parts not exceeding m, and greater than unity. These are enumerated by

$$\frac{x^{\frac{1}{2}m(m+1)}}{(1-x^2)(1-x^3)\dots(1-x^m)}.$$

Summing over m, and using Euler's theorem multiplied by $1-x$, we have

$$\Sigma p_s(n) . x^n = (1-x)(1+x)(1+x^2)(1+x^3)\dots,$$

and the proof may be completed as before, the use of the lemma being avoided.

Fourth proof of Theorem 1: A partition into unequal parts of which the two greatest differ by unity may be regarded as a partition into a part $2r+3$, representing the two largest parts taken together, and unequal parts not exceeding r. These are enumerated by $(1+x)(1+x^2)\dots(1+x^r)x^{2r+3}$. Summing over r, the generating function for $p_s(n)$ is

$$\begin{aligned}
&1 + x^3 + (1+x)x^5 + (1+x)(1+x^2)x^7 + (1+x)(1+x^2)(1+x^3)x^9 + \dots\\
&= (1+x)[1 - x + x^2 + x^5 + (1+x^2)x^7 + (1+x^2)(1+x^3)x^9 + \dots]\\
&= (1+x)(1+x^2)[1 - x(1-x^2) + x^7 + (1+x^3)x^9 + \dots]\\
&= (1+x)(1+x^2)(1+x^3)[1 - x(1-x^3) + x^9 + (1+x^4)x^{11} + \dots]\\
&= \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\\
&= (1+x)(1+x^2)\dots(1+x^n)[1 - x(1-x^n) + x^{2n+3} + \dots
\end{aligned}$$

Letting $n \to \infty$, we have $(1-x)(1+x)(1+x^2)(1+x^3)\ldots$, and the proof may be completed as before.

First proof of Theorem 2: This is the special case $m_1 = 0$ of the second proof above. This gives a (1, 1) correspondence between the partitions of n into odd parts which are not unity and those into unequal parts which are not powers of two.

Second proof of Theorem 2: We have seen that the generating function for $p_2(n)$, and hence $p_1(n)$, may be written

$$\begin{aligned}&(1-x)(1+x)(1+x^2)(1+x^3)(1+x^4)\ldots\\&=(1-x^2)(1+x^2)(1+x^3)(1+x^4)(1+x^5)\ldots\\&=(1-x^4)(1+x^3)(1+x^4)(1+x^5)(1+x^6)(1+x^7)\ldots\\&=(1+x^3)(1+x^5)(1+x^6)(1+x^7)(1-x^8)(1+x^9)\ldots\\&=(1+x^3)(1+x^5)(1+x^6)(1+x^7)(1+x^9)(1+x^{10})\ldots,\end{aligned}$$

which is the generating function for $p_3(n)$, the number of partitions of n into unequal parts which are not powers of two.

Readers are invited to discover a combinatorial proof of Theorem 2.

University of Malaya R.K.G.

1911. EARTH. Refer to its four corners since it is round.

GEOMETRICIAN. "Travelling on strange seas of thought—alone . . ."

GOLDEN NUMBER, DOMINICAL LETTER, ETC. Shown on all calendars but nobody knows what they mean.

INFINITESIMAL. Meaning unknown but it has to do with homeopathy.

MATHEMATICS. Dry up the emotions.

MECHANICS. Lower branch of mathematics.

OCTOGENARIAN. Applies to any elderly man.

OMEGA. Second letter of the Greek alphabet, since everybody always says: "The alpha and omega of . . ."

OMNIBUS. Never a seat to be found. Were invented by Louis XIV. "Let me tell you, sir, that I can remember tricycles when they had only three wheels."

PLANETS. All discovered by M. Leverrier.

POPILIUS. Inventor of a kind of circle. (Popilius Laenas was a Roman consul sent on a mission to a Syrian king who used delaying tactics. Popilius brought him to terms by drawing a circle on the ground and refusing to step outside his diagram until he got an answer).

PROBLEM. "Need only be stated to be solved."

PYRAMID. Useless edifice.

SPELLING. Believe it as absolute as mathematics. Useless if you have style.

SQUARING THE CIRCLE. Nobody knows what this is, but shrug your shoulders at any mention of it.—Jacques Barzun, *Flaubert's Dictionary of Accepted Ideas*, 1954 (Max Reinhardt, London). [Per Mr. G. N. Copley.]

A NEW APPROACH TO LIMITS

By Iain T. Adamson

Let us begin by commenting on the familiar classical formulation of the limit concept. If we have a function $f(x)$ of the real variable x, and a given real number a, then we are accustomed to saying:

(A) *The limit of $f(x)$ as x tends to a is the real number L, $\lim_{x\to a} f(x)=L$, if, corresponding to every positive real number ϵ there exists a positive real number δ such that whenever $0<|x-a|<\delta$ we have $|f(x)-L|<\epsilon$.*

In a recent series of articles and books (see, in particular, [3] and [4]), Menger has raised grave doubts about the meaning, if any, which can be attached to the phrase " a function $f(x)$ of the real variable x ". He proposes to talk instead of " a function f whose domain is the set of real numbers ", claiming that on this basis he can develop the calculus without the use of variables at all; his book [3] is a triumphant vindication of this claim. By denying himself the use of the variable x, Menger is no longer able to use the phrase "as x tends to a" in discussing limits; he boldly returns to a usage similar to that of the nineteenth century and says,

(B) *The limit of f at a is L, $\lim_a f=L$, if, corresponding to every positive real number ϵ there exists a positive real number δ such that whenever*

$$0<|x-a|<\delta$$

we have $|f(x)-L|<\epsilon$.

To drop the phrase " as x tends to a " may be held by some to be a tragedy; certainly its introduction, and that of the arrow notation, were hailed as a great improvement. But the word " tends " inevitably conjures up a picture of x (whatever x is) moving towards a; and as Frege pointed out in an early discussion of the meaning of variables and functions [2], movement can take place only in time, while pure mathematics has nothing to do with time.

Consider now the statement (B). In so far as it contains no mention of " variables " nor of " tending " it is certainly to be preferred to (A); but it still suffers from certain psychological disadvantages. It cannot be too strongly emphasised that (B) defines the whole sentence, " the limit of f at a is L ", and not the single term, " the limit of f at a ". Thus, when students are asked to find the limit of some particular function at a given point, they are being asked to find something which has not been defined—and which apparently cannot be defined, since we are told in some cases that " the limit does not exist." The answer made by the conventional analyst to any criticism of this unsatisfactory position is that obviously the students are being asked to guess a number L such that " $\lim_a f=L$ " is a true proposition; if this is so, then it is a pity that conventional textbooks do not make this obvious point explicitly. Once the point is made, however, there appears to be no further objection to the traditional approach except that mentioned above, namely that sometimes the limit does not exist.

To meet this objection we offer now a definition of the term " the limit of f at a " which is applicable to every function f and every point a which is a point of accumulation of the domain of f. This definition is a slight adaptation of that adopted in [1], which is in turn a simplification of the Moore-Smith definition [5]. Although a similar procedure can be applied to give the limits of functions which map sets of points of real n-space into real

m-space, we shall restrict ourselves for simplicity to the case of real-valued functions of sets of real numbers.

We require one or two preliminary definitions. Let a be a real number, δ a positive real number; then we define the *δ-neighbourhood* of a, $N_\delta(a)$, to be the set of real numbers whose distance from a is less than δ, *i.e.* $N_\delta(a)$ is the set of real numbers x such that $|x-a|<\delta$. The *deleted δ-neighbourhood* of a, $N'_\delta(a)$, is obtained by dropping a itself from $N_\delta(a)$, *i.e.* $N'_\delta(a)$ is the set of real numbers x such that $0<|x-a|<\delta$. It is convenient to adjoin to the set of real numbers the two "ideal real numbers" $-\infty$ and $+\infty$; if δ is any positive real number we define the δ-neighbourhood $N_\delta(-\infty)$ to be the set of real numbers x such that $x<-1/\delta$, together with $-\infty$ itself; $N_\delta(+\infty)$ consists of $+\infty$ and all the real numbers $x>1/\delta$. The deleted δ-neighbourhoods $N'_\delta(-\infty)$ and $N'_\delta(+\infty)$ are obtained by dropping $-\infty$ from $N_\delta(-\infty)$ and $+\infty$ from $N_\delta(+\infty)$ respectively. If E is a set of real numbers, a real number a is said to be a *point of accumulation* of E if, for every positive real number δ, $N'_\delta(a)$ contains at least one point of E. It follows from this definition that $-\infty$ $(+\infty)$ is a point of accumulation of every set of real numbers which is unbounded below (above).

Let now E be a set of real numbers, a a point of accumulation of E. Let f be a real-valued function of E. For every positive real number δ we take the set of numbers in E which lie in $N'_\delta(a)$—this set is non-empty, since a is a point of accumulation of E—and form the set of values of f corresponding to these numbers, *i.e.* we form the set $E_\delta=f(E\cap N'_\delta(a))$. Next we form the closure $\mathrm{Cl}\,(E_\delta)$ of this set, *i.e.* we adjoin to it all its points of accumulation (including $-\infty$ and $+\infty$ if appropriate). Then we make the following definition:

(C) *The E-limit of f at a is the set of real numbers common to all these closures; in symbols,*

$$E\text{-}\lim_{a} f=\bigcap_{\delta} \mathrm{Cl}\,(f(E\cap N'_\delta(a))).$$

We illustrate this definition by applying it to two simple cases in which the limit as defined by (A) or (B) "does not exist". First we consider the function f whose domain E is the set of non-zero real numbers and whose values are given by

$$f(x)=\sin\frac{1}{x}\quad\text{for all real numbers } x\neq 0.$$

If δ is any positive real number, $N'_\delta(0)$ is the union of the two open intervals $(-\delta, 0)$ and $(0, \delta)$. For every δ it is clear that $f(E\cap N'_\delta(0))$ is the closed interval $[-1, 1]$, whence it follows at once that $E\text{-}\lim_0 f$ is this whole closed interval. Again let E be the set of non-zero real numbers, and let f be the function given by

$$f(x)=\frac{1}{x}\quad\text{for all real numbers } x\neq 0.$$

In this case $E_\delta=f(E\cap N'_\delta(0))$ is the union of the two open intervals

$$(-\infty, -1/\delta) \text{ and } (1/\delta, +\infty);$$

hence $\mathrm{Cl}\,(E_\delta)$ is the union of the two closed intervals $[-\infty, -1/\delta]$ and $[1/\delta, +\infty]$. Consequently $E\text{-}\lim_0 f$ is the set consisting of the two ideal real numbers $-\infty$ and $+\infty$. This example illustrates also the importance of taking the domain E into account when we form the limit; if, for example, E_1 is the set of positive real numbers, then $E_1\text{-}\lim_0 f$ consists of $+\infty$ alone.

We now prove a result familiar in the classical context.

Theorem. *Let E be a set of real numbers unbounded above. If f is an increasing function of E then $E\text{-}\lim_{+\infty} f$ contains either a single real number or $+\infty$.*

Proof. Let δ be any positive real number. Then $E_\delta = f(E \cap N'_\delta(+\infty))$ is the set of values $f(x)$ corresponding to all real numbers x in E such that $x > 1/\delta$. Since f is an increasing function, all these sets E_δ have the same least upper bound L, which may be either a real number or $+\infty$. Clearly L belongs to the closure of every E_δ and hence to the set $E\text{-}\lim_{+\infty} f$.

We claim that no other real number belongs to $E\text{-}\lim_{+\infty} f$. Clearly no real number greater than L can belong to $E\text{-}\lim_{+\infty} f$; so let M be a real number less than L. Then M is not the least upper bound of any of the sets E_δ, and hence there is a positive real number x_1 in E such that $f(x_1) > M$. If $\delta_1 = 1/x_1$, then for every real number x in $E \cap N'_{\delta_1}(-\infty)$ we have $x > x_1$ and hence $f(x) > f(x_1) > M$. Thus M does not belong to $\mathrm{Cl}\,(E_{\delta_1})$.

From this discussion it follows that $E\text{-}\lim_{+\infty} f$ consists of L alone.

Let now $\bar{R}$ be the extended real line, consisting of all the ordinary real numbers together with the two ideal real numbers $-\infty$ and $+\infty$. Using the theorem we have just established we can go on to prove the

Heine-Borel Theorem. *Let F be any closed subset of $\bar{R}$, possibly $\bar{R}$ itself. From any family of open sets which covers F we can extract a finite number of these open sets which already cover F.*

(When we say that a family of sets covers F we mean that every point of F lies in at least one of the sets of the family. We remark also that included among the open sets of $\bar{R}$ are sets of the form $[-\infty, a)$ and $(a, +\infty]$.)

The Heine-Borel Theorem has the following useful consequence.

Corollary. *If a family of closed sets in $\bar{R}$ has the property that every finite subfamily has a non-empty intersection, then the whole family has a non-empty intersection.*

Proof. Consider the family of complements of the closed sets—these complements are of course open sets. By hypothesis, no finite subfamily of these open sets covers $\bar{R}$; hence, by the Theorem, the family itself cannot cover $\bar{R}$. By taking complements again the desired result follows.

Using this Corollary we can assert

Theorem. *For every function f and every point of accumulation a of its domain E, $E\text{-}\lim_{a} f$ is non-empty.*

Proof. According to the Corollary we have only to show that any finite collection of the closed sets $\mathrm{Cl}\,(E_\delta)$ has a non-empty intersection. Let $\mathrm{Cl}\,(E_{\delta_1})$, $\mathrm{Cl}\,(E_{\delta_2})$, ..., $\mathrm{Cl}\,(E_{\delta_n})$ be such a finite collection. Let $\delta_1 = \min(\delta_1, \ldots, \delta_n)$. Then clearly $\mathrm{Cl}\,(E_{\delta_1})$ is the intersection of the sets in this collection.

If $E\text{-}\lim_{a} f$ consists of a single real number or $-\infty$ or $+\infty$ we shall say that f is *convergent* at a. It may be objected that the function f of the set of natural numbers given by $f(n) = n$, *i.e.* the sequence 1, 2, 3, ..., cannot properly be called convergent at $+\infty$. To the present writer it seems desirable as far as possible to treat $-\infty$ and $+\infty$ on the same footing as the ordinary real numbers; but those who disagree can say, of course, that if $E\text{-}\lim_{a} f$

contains only $+\infty$, then f diverges to $+\infty$ at a. There is, however, a price that must be paid for thus following convention—it is that separate statements for convergence and divergence to $-\infty$ or $+\infty$ must be given for theorems like the following, which shows that when f is convergent at a the definition (C) and the classical definition (B) are equivalent.

Theorem. *Let E be a set of real numbers, a a point of accumulation of E, L a real number or $-\infty$ or $+\infty$. Let f be a real-valued function of E. Then the following conditions are equivalent:*

1. *f is convergent at a with E-limit L;*
2. *corresponding to every positive real number ϵ there exists a positive real number δ such that for all points x of E in the deleted δ-neighbourhood of a, $f(x)$ lies in the ϵ-neighbourhood of L.*

Proof. Suppose condition 1 holds.

If condition 2 does not hold, then there exists a positive real number ϵ such that, for every positive real number δ, $E_\delta = f(E \cap N'_\delta(a))$ contains a real number outside $N_\epsilon(L)$; *a fortiori*, $\mathrm{Cl}(E_\delta)$ contains a real number outside $N_\epsilon(L)$. The closed sets F_δ consisting of those real numbers which belong to $\mathrm{Cl}(E_\delta)$ and do not belong to $N_\epsilon(L)$ are thus non-empty; any finite collection of them has a non-empty intersection, and hence the whole collection of sets F has a non-empty intersection. That is to say, $E\text{-}\lim_a f = \cap \mathrm{Cl}(E_\delta)$ contains a point outside $N_\epsilon(L)$. But this is a contradiction since, by hypothesis, $E\text{-}\lim_a f$ consists of L alone.

Thus condition 1 implies condition 2.

Conversely, suppose condition 2 holds.

Then, corresponding to every positive real number ϵ there exists a positive real number $\delta(\epsilon)$ such that $E_{\delta(\epsilon)} = f(E \cap N'_{\delta(\epsilon)}(a))$ is contained in $N_\epsilon(L)$. Thus $\mathrm{Cl}(E_{\delta(\epsilon)})$ is contained in $\mathrm{Cl}(N_\epsilon(L))$. Now $E\text{-}\lim_a f$, which is the intersection over *all* positive real numbers δ of the sets $\mathrm{Cl}(E_\delta)$ is contained in the intersection over all positive real numbers ϵ of the sets $\mathrm{Cl}(E_{\delta(\epsilon)})$, and hence in the intersection of all the sets $\mathrm{Cl}(N_\epsilon(L))$. But this last intersection consists simply of L itself, and since $E\text{-}\lim_a f$ is non-empty, it follows that $E\text{-}\lim_a f$ consists of L also.

Thus condition 2 implies condition 1.

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SUMS OF POWERS OF THE INTEGERS

By R. V. Parker

Recent Notes in the *Mathematical Gazette* [1] have suggested directly or by implication the surprise felt by some students of mathematics when Σx^3 turns out to be $(\Sigma x)^2$. Perhaps the results appear less surprising if they are seen, not in isolation, but as two particular cases of summation formulae for *odd* powers of the integers expressed in powers of $\{x(x+1)\}$.

The fact that the sums of *odd* powers of the integers are expressible in powers of $\{x(x+1)\}$ can be used as a demonstration of the distinctive pattern of the *difference tables* of these sums, which facilitates their speedy calculation.

Summation formulae and difference tables for *even* powers of the integers are seen to be somewhat similar to those for *odd* powers.

Let $\Sigma u_x = u_1 + u_2 + u_3 + \ldots + u_x$.

$\nabla u_x = u_x - u_{x-1}$.

$\binom{n}{r}$ = the binomial coefficient $\dfrac{n(n-1)\ldots(n-r+1)}{r!}$.

Then
$$\nabla x^n(x+1)^n = x^n(x+1)^n - (x-1)^n \cdot x^n$$
$$= x^n\left\{x^n + n\cdot x^{n-1} + \binom{n}{2}x^{n-2} + \binom{n}{3}x^{n-3} + \ldots\right.$$
$$\left. - x^n + n\cdot x^{n-1} - \binom{n}{2}x^{n-2} + \binom{n}{3}x^{n-3} - \ldots\right\}$$
$$= x^n\left\{2n\cdot x^{n-1} + 2\binom{n}{3}x^{n-3} + 2\binom{n}{5}x^{n-5} + \ldots\right\}$$
$$= 2\left\{n\cdot x^{2n-1} + \binom{n}{3}x^{2n-3} + \binom{n}{5}x^{2n-5} + \ldots\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\text{(i)}$$

$$\Sigma\nabla x^n(x+1)^n = (1^n\cdot 2^n - 0^n\cdot 1^n) + (2^n\cdot 3^n - 1^n\cdot 2^n) + \ldots$$
$$\ldots + \{(x-1)^n\cdot x^n - (x-2)^n(x-1)^n\}$$
$$+ \{x^n(x+1)^n - (x-1)^n\cdot x^n\}$$
$$= x^n(x+1)^n. \quad \ldots\ldots\ldots\ldots\ldots\ldots\text{(ii)}$$

From (i) and (ii) we obtain:
$$\sum\left\{n\cdot x^{2n-1} + \binom{n}{3}x^{2n-3} + \binom{n}{5}x^{2n-5} + \ldots\right\} = \tfrac{1}{2}\cdot x^n(x+1)^n, \quad \ldots\ldots\ldots\text{(iii)}$$

from which it is clear that the sum of any *odd* power of the integers (and only the sums of odd powers of the integers) may be expressed as powers of $\{x(x+1)\}$. For example, putting $n = 1$, 2 and 3 we obtain:

$n=1$. $\Sigma x = \tfrac{1}{2}\cdot x(x+1)$.

$n=2$. $2\Sigma x^3 = \tfrac{1}{2}\cdot x^2(x+1)^2$, whence $\Sigma x^3 = \tfrac{1}{4}\cdot x^2(x+1)^2$.

$n=3$. $\Sigma(3x^5 + x^3) = \tfrac{1}{2}\cdot x^3(x+1)^3$
$$3\Sigma x^5 = \tfrac{1}{2}\cdot x^3(x+1)^3 - \Sigma x^3$$
$$= \tfrac{1}{2}\cdot x^3(x+1)^3 - \tfrac{1}{4}\cdot x^2(x+1)^2$$
$$\Sigma x^5 = \frac{x^3}{6}(x+1)^3 - \frac{x^2}{12}(x+1)^2.$$

For Σx^n (n odd) the number of terms will be $\frac{1}{2}(n-1)$, and the indices will decrease from $\frac{1}{2}(n+1)$ down to 2, the term being alternately positive and negative. The first coefficient will be $\dfrac{1}{n+1}$.

$\Sigma x^2 = \dfrac{x}{6}(x+1)(2x+1)$, and all Σx^n (n even) is divisible by the factor $2x+1$.

$$\nabla x^n(x+1)^n(2x+1) = 2\left[\left\{\binom{n}{0}+2\binom{n}{1}\right\}x^{2n} + \left\{\binom{n}{2}+2\binom{n}{3}\right\}x^{2n-2}+\dots\right]$$

whence, since $[x^n(x+1)^n(2x+1)]_{x=0}=0$, we have :

$$\Sigma\nabla x^n(x+1)^n(2x+1) = x^n(x+1)^n(2x+1)$$

i.e.,

$$\sum\left[\left\{\binom{n}{0}+2\binom{n}{1}\right\}x^{2n} + \left\{\binom{n}{2}+2\binom{n}{3}\right\}x^{2n-2} + \left\{\binom{n}{4}+2\binom{n}{5}\right\}x^{2n-4}+\dots\right]$$
$$=\tfrac{1}{2}\,.\,x^n(x+1)^n(2x+1). \qquad \text{(iv)}$$

which shows that the sum of any *even* power of the integers (and only the sums of even powers of the integers) may be expressed as powers of $\{x(x+1)\}$, each power with the additional factor $2x+1$.

For $n=1$ and 2, we obtain :

$n=1$. $\quad \Sigma(1+2)x^2 = 3\,\Sigma x^2 = \frac{1}{2}\,.\,x(x+1)(2x+1)$

whence $\Sigma x^2 = \frac{1}{6}x(x+1)(2x+1)$.

$n=2$. $\quad \Sigma\{(1+4)x^4+(1+0)x^2\} = \Sigma(5x^4+x^2) = \frac{1}{2}\,.\,x^2(x+1)^2(2x+1)$

$$5\,\Sigma x^4 = \tfrac{1}{2}\,.\,x^2(x+1)^2(2x+1) - \Sigma x^2$$
$$= \tfrac{1}{2}\,.\,x^2(x+1)^2(2x+1) - \tfrac{1}{6}x(x+1)(2x+1)$$
$$\Sigma x^4 = \tfrac{1}{10}x^2(x+1)^2(2x+1) - \tfrac{1}{30}x(x+1)(2x+1).$$

For Σx^n (n even) the number of terms will be $\frac{1}{2}n$, and the indices of $\{x(x+1)\}$ will decrease from $\frac{1}{2}n$ down to 1, the terms being alternately positive and negative. The first coefficient will be $\dfrac{1}{2(n+1)}$.

The expression of sums of odd powers of the integers in powers of $\{x(x+1)\}$ and of even powers of the integers as $(2x+1)$ times powers of $\{x(x+1)\}$ has been demonstrated by a different method by *Piza* [2].

We now wish to demonstrate that the function representing Σx^n (n odd) has a *difference table* of a distinctive pattern which facilitates calculation.

$$\Sigma x^n \ (n \text{ odd}) = \sum_{t=0}^{n-2} c_t\,.\,x^{n-t}(x+1)^{n-t}, \text{ where } c_0, c_1, \dots, c_{n-2} \text{ are constants.}$$

Let u_x represent the function for Σx^n (n odd), and put $-x-1$ for x. Then :

$$u_{-x-1} = \sum_{t=0}^{n-2} c_t\,.\,(-x-1)^{n-t}(-x)^{n-t}$$
$$= \sum_{t=0}^{n-2} c_t(-1)^{n-t}(x+1)^{n-t}(-1)^{n-t}\,.\,x^{n-t}$$
$$= \sum_{t=0}^{n-2} c_t\,.\,x^{n-t}(x+1)^{n-t}.$$

Hence $$u_x = u_{-x-1}. \qquad \text{(v)}$$

Let
$$\Delta u_x = u_{x+1} - u_x,$$
$$\Delta^2 u_x = \Delta u_{x+1} - \Delta u_x = u_{x+2} - 2u_{x+1} + u_x,$$
$$\Delta^r u_x = \sum_{t=0}^{r} (-1)^t \binom{r}{t} u_{x+r-t}. \qquad \text{(vi)}$$

In (vi) put $2x-1$ for r and $-x$ for x. We then have:
$$\Delta^{2x-1} u_{-x} = \sum_{t=0}^{2x-1} (-1)^t \binom{2x-1}{t} u_{x-t-1}$$
$$= \binom{2x-1}{0} u_{x-1} - \binom{2x-1}{1} u_{x-2} + \dots + \binom{2x-1}{2x-2} u_{-x+1} - \binom{2x-1}{2x-1} u_{-x}$$

giving an even number of terms which may be paired as under:
$$\binom{2x-1}{0} u_{x-1} - \binom{2x-1}{1} u_{x-2} + \binom{2x-1}{2} u_{x-3} - \binom{2x-1}{3} u_{x-4} + \dots \pm \binom{2x-1}{x-1} u_0$$
$$-\binom{2x-1}{2x-1} u_{-x} + \binom{2x-1}{2x-2} u_{-x+1} - \binom{2x-1}{2x-3} u_{-x+2} + \binom{2x-1}{2x-4} u_{-x+3}$$
$$\dots \mp \binom{2x-1}{x} u_{-1}$$

The binomial identity $\binom{n}{r} = \binom{n}{n-r}$ and the equality (v) above show that the sum of the above two rows of terms is zero, in the case of the function for Σx^n (n odd).

We thus have, if u_x is the function Σx^n (n odd):
$$\Delta^{2x-1} u_{-x} = 0,$$
whence, since $$\Delta^r u_x = \Delta^{r-1} u_{x+1} - \Delta^{r-1} u_x,$$
we have:
$$\Delta^{2x} u_{-x} = \Delta^{2x-1} u_{-x+1} - \Delta^{2x-1} u_{-x}$$
and $$\Delta^{2x} u_{-x} = \Delta^{2x-1} u_{-x+1}. \qquad \text{(vii)}$$

We demonstrate this equality of differences in the following table, where u_x stands for the function which represents Σx^5, and values are given for u_x for unit intervals of the argument from $x = -3$ to $x = 3$.

x	u_x	Δu_x	$\Delta^2 u_x$	$\Delta^3 u_x$	$\Delta^4 u_x$	$\Delta^5 u_x$	$\Delta^6 u_x$
-3	33						
		-32					
-2	1		31				
		-1		-30			
-1	0		1		30		
		0		0		0	
0	0		1		30		120
		1		30		120	
1	1		31		150		
		32		180			
2	33		211				
		243					
3	276						

The equalities shown in (vii) are linked by the "zig-zag" line, and are seen to be the differences for *Gauss's forward formula for equal intervals* [3], which may be expressed as :

$$u_x = u_0 + \Delta u_0 \binom{x}{1} + \Delta^2 u_{-1} \binom{x}{2} + \Delta^3 u_{-1} \binom{x+1}{3} + \Delta^4 u_{-2} \binom{x+1}{4} + \dots .$$

We thus have :

$$\Sigma x^n \ (n \text{ odd}) = c_0 \left\{ \binom{x}{1} + \binom{x}{2} \right\} + c_1 \left\{ \binom{x+1}{3} + \binom{x+1}{4} \right\}$$
$$+ c_2 \left\{ \binom{x+2}{5} + \binom{x+2}{6} \right\} + \dots$$
$$= c_0 \binom{x+1}{2} + c_1 \binom{x+2}{4} + c_2 \binom{x+3}{6} + \dots$$

where $c_0 = 1$, the last constant equals $n!$, and the number of terms is $\frac{1}{2}(n+1)$.

Example. Evaluate Σx^5.

There will be $\frac{1}{2}(5+1) = 3$ terms. The first and last coefficients will be 1 and 120 respectively. We therefore require to find the middle coefficient, for which we only need $2^5 = 32$.

0   1   30   120

    1   31   30   120

32

$$\text{whence } \Sigma x^5 = \binom{x+1}{2} + 30 \binom{x+2}{4} + 120 \binom{x+3}{6} .$$

The difference table for Σx^n (n even) can similarly be shown to conform to the pattern indicated in the following example for Σx^4.

x	u_x	Δu_x	$\Delta^2 u_x$	$\Delta^3 u_x$	$\Delta^4 u_x$	$\Delta^5 u_x$
-3	-17					
		16				
-2	-1		-15			
		1		14		
-1	0		-1		-12	
		0		2		24
0	0		1		12	
		1		14		
1	1		15			
		16				
2	17					

The differences joined by the "zig-zag" line are seen to be the differences for *Gauss's backward formula for equal intervals* [4] which may be given as :

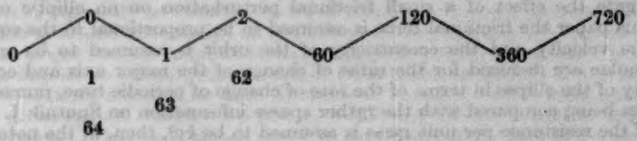
$$u_x = u_0 + \Delta u_{-1} \binom{x}{1} + \Delta^2 u_{-1} \binom{x+1}{2} + \Delta^3 u_{-2} \binom{x+1}{3} + \Delta^4 u_{-2} \binom{x+2}{4} + \dots$$

We thus have :

$$\Sigma x^n \ (n \text{ even}) = c_0 \binom{x+1}{2} + 2 \,.\, c_0 \binom{x+1}{3} + c_1 \binom{x+2}{4} + 2 \,.\, c_1 \binom{x+2}{5} + \dots$$

Example. Evaluate Σx^6.

The first coefficient will be 1, the second 2, the last $n!$ and the last but one $\frac{1}{2} \,.\, n!$. We require to find the third. The fourth will be twice the third. We thus only need $2^6 = 64$.



$$\text{Whence } \Sigma x^6 = \binom{x+1}{2} + 2\binom{x+1}{3} + 60\binom{x+2}{4} + 120\binom{x+2}{5}$$

$$+ 360\binom{x+3}{6} + 720\binom{x+3}{7}.$$

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1912. Mathematics go on Parade.

The Great Hall of Eltham Palace, where Kings and Parliament have sat, rocked to martial tread yesterday afternoon.

A squad of young Guardsmen, ably seconded by a squad of boys from the Duke of York's Royal Military School, were demonstrating " Mathematics on Parade ".

This is a system designed to make arithmetical processes visually comprehensible, first by parade movements and afterwards by graphs.

Its originator, Mr. W. G. Bass, director of an engineering firm, and its applier, Mrs. O. S. Dowty, headmistress of Headington Infant School, Oxford, watched yesterday's demonstration.

With other educationists they saw how the Royal Army Education Corps is applying the system to backward soldiers.

In tribute to the system's efficacy I need only say that by the time my observer came away he was able to subtract pounds, shillings and pence, reading from left to right. And that, I believe, is what is known as higher mathematics.—*Daily Telegraph & Morning Post*, July 12, 1955. [Per Mr. B. M. Brown.]

THE EFFECT OF FRICTION ON ELLIPTIC ORBITS

By D. G. Parkyn

1. *Introduction*

It forms an interesting and topical example in particle dynamics to investigate the effect of a small frictional perturbation on an elliptic orbit. In this paper the frictional force is assumed to be proportional to the square of the velocity and the eccentricity of the orbit is assumed to be small. Formulae are deduced for the rates of change of the major axis and eccentricity of the ellipse in terms of the rate of change of periodic time, numerical values being compared with the rather sparse information on Sputnik I.

2. If the resistance per unit mass is assumed to be kv^2, then, in the notation of Ramsey (*Dynamics*, Vol. II)

$$\dot{a} = -\frac{2ka^2v^3}{\mu}$$

and

$$\dot{e} = -\frac{2ka(1-e^2)}{e}\left(\frac{1}{r}-\frac{1}{a}\right)v.$$

Since $\dot{a}$ and $\dot{e}$ vary appreciably during a revolution, it is convenient to determine the changes in a and e, (Δa, Δe) over one complete revolution. Thus, using the normal relations for elliptic orbits, we obtain

$$\Delta a = -2ka^2\int_0^{2\pi}\frac{(1+2e\cos\theta+e^2)^{3/2}}{(1+e\cos\theta)^2}\,d\theta,$$

$$\Delta e = -2ka(1-e^2)\int_0^{2\pi}\frac{(1+2e\cos\theta+e^2)^{1/2}\,(e+\cos\theta)}{(1+e\cos\theta)^2}\,d\theta.$$

Let

$$A = \int_0^{2\pi}\frac{(1+2e\cos\theta+e^2)^{1/2}}{(1+e\cos\theta)^2}\,d\theta$$

and

$$B = \int_0^{2\pi}\frac{(1+2e\cos\theta+e^2)^{1/2}\cos\theta}{(1+e\cos\theta)^2}\,d\theta,$$

then

$$\Delta a = -2ka^2\{(1+e^2)A+2eB\}$$

and

$$\Delta e = -2ka(1-e^2)\{eA+B\}.$$

To evaluate the integrals A and B expand them as power series in e and neglect terms above the square :

$$A = \int_0^{2\pi}\left\{1-e\cos\theta+\frac{e^2}{2}(1+\cos^2\theta)\right\}d\theta$$

$$=\frac{\pi}{2}(4+3e^2).$$

$$B = \int_0^{2\pi}\left\{1-e\cos\theta+\frac{e^2}{2}(1+\cos^2\theta)\right\}\cos\theta\,d\theta$$

$$= -\pi e.$$

Whence $\Delta a = -\pi ka^2(4+3e^2)$

and $\Delta e = -2\pi kae.$

Since $T = 2\pi\sqrt{\dfrac{a^3}{\mu}}$ we have that $\Delta a = n\dfrac{T\Delta T}{6\pi^2a^2}$.

and hence $$\Delta e = \frac{\mu T \Delta T}{24\pi^2 a^2}(1 - \tfrac{3}{4}e^2).$$

3. *Numerical Results*

It is possible, assuming constant k, to integrate these relations and develop a, e and T as functions of the number of revolutions, n. However, the rates of change are so small that it seems adequate to assume a linear dependence upon n, especially as variation in k will eventually become predominant.

The available information on the initial orbit of Sputnik I is that its maximum height was 630 miles and that its apparent periodic time was 1·59 hours. This implies an actual periodic time of 1·696 hours, calculated on the number of revolutions attained on 13th November. Hence $a = 4{,}507$ miles and using the given apsidal distance we find $e = 0{\cdot}027$, and the other apsidal height as 384 miles. It has been stated that T is decreasing at the rate of 2·9 secs. per 24 hours.

From this we find

$$\Delta a = -0{\cdot}1103 \text{ miles}$$

and $$\Delta e = -3{\cdot}04 \times 10^{-7}.$$

This would imply that the satellite would fall 120 miles in about 83 days, on the assumption of a constant density. The assumption that higher powers of e than the square may be neglected is amply justified for the given orbit, the only questionable assumption being the constancy of the density.

The statistically predicted density is of the order of 10^{-14} gms./c.c. at 100 miles and falls to 10^{-42} gms./c.c. at 560 miles, obviously inadequate to explain the stated rate of change of T. In fact, on the Newtonian resistance law—reasonably accurate for high speeds—we can estimate the required density as about 3×10^{-14} gms./c.c. One possible solution of the contradiction is in the presence of dust matter in the solar system, a reasonable supposition from observation of other galaxies. In this case one would expect a roughly constant density above the " statistical atmosphere ".

The resultant rate of fall then disagrees with the maximum apsidal distance published by the Russians on 11th November, 1957, of 503 miles, a fall of 127 miles in 39 days. In other words, the available information is inconsistent. The periodic time is probably correct and is capable of a simple check. The rate of decrease of periodic time may be compared with a Cambridge estimate of 1·8 secs. per 24 hrs., which would anyway give a slower rate of fall. An average height of about 500 miles is consistent with the periodic time, but it seems likely that the " maximum distances " are in error.

The validity of the original 630 miles would give the satellite a life of some 200 days, and of the second, a total life of 155 days. The average of these would allow a life of about 6 months—to mid-April, 1958.

University of Natal D.G.P.

Additional Note

The paradoxical behaviour of Sputnik I, which travelled faster through the effect of friction, can also be demonstrated mathematically in the following elementary way.

Consider a particle describing a circular orbit of radius r_0 with speed v_0 under a central attraction κ/r^2. Then $v_0^2/r_0 = \kappa/r_0^2$ and $\kappa = r_0 v_0^2$. Now suppose that v and r change slowly with time under the influence of a tangential retardation λv_0. To a first approximation

$$v = r\dot\theta \quad \text{and} \quad \ddot r - r\dot\theta^2 = -\kappa/r^2$$
$$r\ddot\theta + 2\dot r\dot\theta = -\lambda v_0.$$

Now suppose that after a short time t, $r=r_0(1+\alpha t)$, $v=v_0(1+\beta t)$, $\dot\theta=\dot\theta_0(1+\gamma t)$. Then the above equations give, neglecting t^2, and equating coefficients of t:

$$\beta=\alpha+\gamma$$

$$\alpha+2\gamma=-2\alpha$$

$$\gamma+2\alpha=-\lambda$$

from which $\alpha=-2\lambda$, $\beta=\lambda$, $\gamma=3\lambda$.

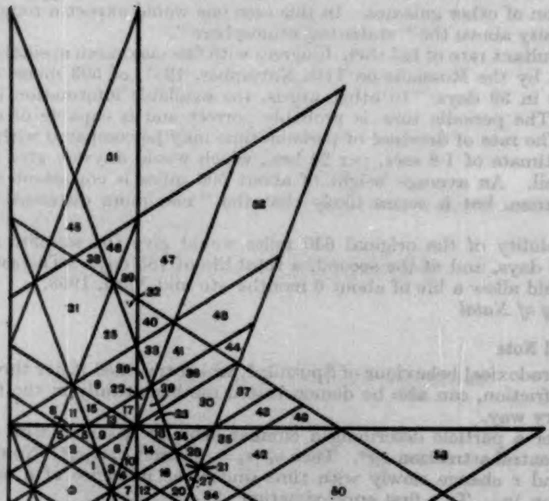
Finally $r=r_0(1-2\lambda t)$, $v=v_0(1+\lambda t)$, $\dot\theta=\dot\theta_0(1+3\lambda t)$; showing that while r decreases with time, v and $\dot\theta$ both actually increase.

H.M.C.

RHOMBIC TRIACONTAHEDRA

By John D. Ede

The process of producing the facial planes of a polyhedron to form stellated polyhedra is well known. When applied to the icosahedron it leads to a series which is limited to eight solids if we make the condition that every face of each must be covered in forming the next of the series. If this condition is dropped a large number of combinations of the parts of these eight solids is possible. The total number of solids, limited by symmetry considerations, is 59, and they are all illustrated in *The 59 Icosahedra* by Coxeter, Du Val, Flather, and Petrie. The corresponding basic series of solids starting with the Rhombic Triacontahedron numbers 13, and the total number of combinations is very great. Details of the 13 solids are given in the table below.



Rhombic Triacontahedra

Power	Solid	Faces	Solid cells added	Faces of solid cells Inward	Faces of solid cells Outward	No. of units
0	*A*	0	*a*		0	1
$\leqslant 1$	*B*	1	*b*	0(1)	1(4)	30
$\leqslant 2$	*C*	2, 3	*c*	1(2)	2(1), 3(2)	60
$\leqslant 3$	*D*	4, 5, 6	d_1	3(2)	4(2), 5(2)	60
			d_2	2(3)	6(6)	20
$\leqslant 4$	*E*	7, 8, 9, 10	e_1	4(2)	7(1), 8(1)	60
			$e_2(2)$	5(1), 6(1)	9(1), 10(1)	60, 60
$\leqslant 5$	*F*	11, 12, 13, 14	f_1	7(5)	11(5)	12
			f_2	8(1), 10(2)	12(1), 13(2)	60
			f_3	9(4)	14(4)	30
$\leqslant 6$	*G*	15, 16, 17, 18	g_1	11(1), 12(1)	15(2)	60
			$g_2(2)$	13(1), 14(1)	16(1), 17(1), 18(1)	60, 60
$\leqslant 7$	*H*	19, 20, 21	h_1	17(2)	20(1), 23(2)	60
		22, 23, 24	h_2	15(2), 16(2)	19(2), 24(2)	60
			h_3	18(2)	21(2), 22(2)	60
$\leqslant 8$	*I*	25, 26, 27	i_1	19(2), 20(1)	25(2), 30(2)	60
		28, 29, 30	$i_2(2)$	21(1), 24(1)	26(1), 28(1)	60, 60
			$i_3(2)$	22(1), 23(1)	27(1), 29(1)	60, 60
$\leqslant 9$	*J*	31, 32, 33	j_1	25(2)	31(1), 37(2)	60
		34, 35, 36, 37	j_2	28(4)	33(4)	30
			j_3	29(6)	34(3)	20
			$j_4(2)$	26(1), 27(1), 30(1)	32(1), 35(1), 36(1)	60, 60
$\leqslant 10$	*K*	38, 39, 40	k_1	34(1), 36(2)	39(2), 42(2)	60
		41, 42, 43, 44	$k_2(2)$	32(1), 37(1)	38(1), 43(1), 44(1)	60, 60
			$k_3(2)$	33(1), 35(1)	40(1), 41(1)	60, 60
$\leqslant 11$	*L*	45, 46, 47	l_1	40(2), 43(2)	45(1), 48(2)	60
		48, 49, 50	l_2	41(2), 42(2)	47(2), 50(1)	60
			$l_3(2)$	39(1), 44(1)	46(1), 49(1)	60, 60
<12	*M*	51, 52, 53	$m(2)$	47(1), 48(1), 49(1)	51(1), 52(1), 53(1)	60, 60

No attempt has been made to enumerate the others, but details are given of two that are illustrated in *Mathematical Models* by Cundy and Rollett, where they are chosen for their particular properties rather than as members of the triacontahedron series.

The diagram shows the lines in which the plane of one face of the rhombic triacontahedron is cut by the planes of 28 other faces (the remaining face being parallel). To save space only part of the network is shown ; the rest is symmetrical. The large number of parallel lines on the diagram makes it a comparatively simple matter to work out all the lengths in terms of the edge of the original triacontahedron, knowing that its faces have diagonals in the golden ratio.

The solids are mostly complex ones, and making models is a lengthy business. The net is conveniently made up from a number of identical pieces, copied by pricking through, but most of the models will need strengthening with internal struts to keep them rigid, or else should be built on a foundation of some suitable simpler and rigid model. Many of the solids not in the main series are simpler, and this makes them both easier to make and, on the whole, more elegant.

NOTES ON TABLE

Column 1. The method of enumerating the solids is based on one of those in *The 59 Icosahedra* by Coxeter, Du Val, Flather, and Petrie. The term " Power " is defined there, though probably the meaning is clear from the context of the table.

Column 2. Solid A is the Rhombic Triacontahedron. Solid E is the compound of 5 cubes.

Column 3. Numbers in the table refer to the diagram.

Column 4. (2) means that the solid cell exists in two enantiomorphous forms.

Columns 5 and 6. Numbers in brackets show the total of each numbered face in each solid cell : *e.g.* solid cell b has one of face No. 0 facing inwards and four of face No. 1 facing outwards. The face No. 0 joins it to A to make B.

Column 7. 60, 60 means that there are 60 units of each of the two enantiomorphous forms.

Some points with powers of 10, 11, 12 have an infinite locus. There is no parallel to this situation in the case of the 59 Icosahedra.

Two Rhombic Triacontahedra (though not solids of the main sequence A–M) are illustrated in *Mathematical Models* by Cundy and Rollett.

Small stellated triacontrahedron	Faces 2, 5, 8, 11 Solid C plus cells d_1, e_1, f_1
Great stellated triacontahedron	Faces 15, 16, 20, 21, 27, 34 Solid G plus cells h_1, h_2, i_2, j_2

Kingswood School J.D.E.

THE SOLUTION OF QUADRATIC EQUATIONS

By K. S. Snell

There are four methods which are normally taught in schools for solving quadratic equations :

(i) By factorisation.
(ii) By the use of graphs.
(iii) By completing the square.
(iv) By the use of the formula.

Two principles which should always be observed in teaching are :

(*a*) A new topic should be led up to, incidentally, appreciably before it becomes the central objective of a group of lessons.

(*b*) Applications of any topic should always be used to drive home the value of the topic and to give additional examples to consolidate a pupil's knowledge and ability to use any method.

Of the methods quoted above the first two use these principles, while the fourth is the final method which all pupils should know and retain for future use.

Bearing this in mind I suggest that the order of teaching should be as I have given, though the first two are independent and hence their order is immaterial. There are also geometrical methods for solving equations which are interesting and use my second principle, but are not retained by the pupil as standard methods.

Factorisation is welcomed by a teacher as a subject which requires drill, and with which most pupils can be reasonably successful, provided the examples are not unduly hard. Hence there is a tendency to make skill in factorisation a final objective rather than a means to facilitate other mathematical working. It is the link between topics which is to be emphasised, and factors should be used to simplify arithmetical working, especially in mensuration, to help in adding or multiplying fractions, and to provide a means of solving quadratic equations. It is this last application with which we are here concerned, and it enables a quadratic equation to be split up into two simple equations, of a type with which the pupil is familiar. There should be preliminary discussion to answer the question : " Tell me any two numbers whose product is zero? " Then in the solution of an equation like

$$5x^2 - 2x - 3 = 0,$$

an essential step for a pupil to write is

$$5x + 3 = 0 \quad or \quad x - 1 = 0,$$

each simple equation then being solved independently. I still find some boys who say that for an equation

$$(x-3)(x+2) = 0,$$

the answer is obtained by changing the sign of the numbers, i.e. $x = +3$ or -2, a rule which obscures the principle used and can so easily lead to mistakes such as giving $x = -3$ as a solution in the first example above.

I remember being worried often by the introduction given in books to the solution of equations by graphical methods. My difficulty was that the first examples were on the solution of quadratic equations, and boys considered this a clumsy and inaccurate method of attempting something which they could do more quickly and accurately by calculation. Hence I insisted always on making my early examples concerned with cubic equations, a type which a boy could not solve by any other means. But this meant more complicated

graphs than were advisable at the beginning of a subject. The modern stress on the importance of functional thinking and graphical representation has solved my difficulty in a much better way. The representation of linear and quadratic functions by graphs precedes any formal teaching of the solution of quadratic equations. From a graph a boy can pick out maximum or minimum values of a function, the range of values of x for which a function increases or decreases, and the values of x for which the function takes certain specific values. Thus my two principles can both be satisfied in treating graphs of quadratic functions, and the pupil has a visual representation of the fact that quadratic equations may have two, one or no solutions, before he attempts a method for calculating such solutions. It is worth noticing that if the pupil draws an accurate graph of x^2, or uses one from a book, then any equation of the type $x^2=px+q$ can be solved with good accuracy. At this stage a pupil can see that graphical representation can lead to methods of solving any algebraic or trigonometric equation.

I have for long battled with the method of completing the square, with more intelligent boys, in the belief that all who were going on with Calculus would need the method in integration and hence should learn it in their elementary course. To make the method more general I used it for finding maximum and minimum values of quadratic functions. Thus

$$5x^2-2x-3=5(x-\tfrac{1}{5})^2-3\tfrac{1}{5},$$

and hence has a minimum value of $-3\frac{1}{5}$, occurring when $x=\frac{1}{5}$. Also I insisted that at each step of the solution of an equation by completing the square checks could be made, and a boy could really understand what he was doing. The most important check comes after the line

$$x^2+\frac{8x}{3}+\qquad=\frac{2}{3}+\qquad,$$

when a pupil should be taught to write next the following line

$$(x+\tfrac{4}{3})^2=\qquad,$$

and then complete the previous line by multiplying out the latter line. In spite of such checks I have found, to my chagrin, that boys in lower divisions, who had quickly moved to a solution by formula, got more questions right than my own upper division who were completing the square. The latter method provides so many places at which slips can be made, and are made because boys, and I expect girls also, will work each separate step by rule without checking at each stage.

I am thus driven to the conclusion that the fourth method, by the use of the formula, is the one that should be given as the method when the solution of a quadratic equation by calculation becomes the central objective with a pupil. Also it is unwise to precede this by any prolonged use of completing the square because boys (and girls also?) are so conservative that they like to go on using the method they learn first. Happily this does not preclude the early application of methods (i) and (ii) since a boy will realise that he needs another method for calculating accurately and quickly the solution of any quadratic equation. The introductory work would include the solution by pupils of equations like

$$x^2=5,\quad (x+2)^2=7,\quad x^2-6x=3.$$

But then it is possible to obtain the solution of the general equation

$$ax^2+bx+c=0,$$

with the help of the class. But, to their relief, they should not at this stage

be required to reproduce the proof, but only to learn the result, as a formula, and then how to apply it.

It has been suggested that the formula for the solution of the equation

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is easier than that of the more general equation. This is true, but in application an extra step is needed, the division by the coefficient of x^2, and an extra choice is given of expressing the other coefficients in fractions or decimals. This opens the door to other mistakes, such as writing $\frac{2}{3}$ as 0·6 or 0·66. Hence I advocate the use of the more general formula.

May I summarise with the suggestion that quadratic equations should be introduced as an application of factorising, and as an illustration of one use of the graph of a quadratic function. The main work on quadratic equations should be with the general formula, preceded by a short explanatory treatment of completing the square.

Harrow School K.S.S.

CORRESPONDENCE

To the Editor of the *Mathematical Gazette*

DEAR SIR,—There are two matters of historical interest in the A.M.A. book, *The Teaching of Mathematics*, to which I should like to draw attention.

(i) On p. 93 the book implies that to the 1923 M.A. *Report on the Teaching of Geometry* was due the idea of dividing the subject into stages and the modern method of treating the early stages. The book does not point out that the first authoritative statement of these ideas was in the Board of Education Circular No. 711 published in March 1909. (The stages were not quite the same as those suggested in the 1923 report.)

Circular No. 711 suggested that, in all but the last stages the theorems connected with angles at a point, the angles made by a transversal cutting parallel lines, and the congruence theorems should be taken as axioms (or postulates). It gave an admirable description of how those theorems should be led up to.

This laid the foundations for the modern views on the teaching of geometry. It is now well known that Circular No. 711 was mainly, if not entirely, due to W. C. Fletcher, though it was signed by a secretary of the Board of Education.

The failure to mention Circular No. 711 in the A.M.A. book is probably due to the fact that no member of the committee was old enough even to have been at school in 1909.

(ii) The other matter is concerned with the teaching of Elementary Calculus. The book suggests that Calculus was only taught to a few able pupils before the publication in 1944 of the Jeffery report; but the Oxford and Cambridge Joint Examination Board had a paper on Coordinate Geometry and Elementary Calculus for additional mathematics in its School Certificate Examination as early as 1921, though credit in additional mathematics could be obtained without taking that paper. Other examining bodies probably had papers on Elementary Calculus about the same date.

The book speaks of introducing Elementary Calculus " to fifth form pupils ", but before 1913 some schools were already teaching the subject to pupils below the fifth form.

Yours, etc., A. W. SIDDONS

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Yours, etc., A. W. Siddons

To the Editor of the *Mathematical Gazette*

SIR,—At the Oxford Mathematical Conference in April 1957, and again at the Cambridge Colloquium in the summer, it was suggested that a certain amount of Numerical Analysis might usefully be introduced into school work, even if only in an informal way. If there are other teachers whose interest in this proposal is as inhibited by their ignorance of the subject as mine was, may I be permitted to draw their attention to a recent National Physical Laboratory publication, *Modern Computing Methods* (H.M.S.O. Code No. 48-120-16*, price 10s. 6d. net)? I came across it by chance and have nowhere seen it advertised, but it packs into 129 pages a great deal of information about matrices, the numerical solution of linear and polynomial equations; successive differences and their uses in interpolation, integration, and the detection of errors; the solution of ordinary and partial differential equations; relaxation methods; the preparation of mathematical tables; electronic computers and the differential analyser. There is also an extensive bibliography.

Cheltenham College Yours, etc., A. BARTON

To the Editor of the *Mathematical Gazette*

DEAR SIR,—I wish to put forward the following hypothesis with the hope that through the columns of the *Gazette* I may get further information to help " test " it.

" If a boy is good at rugby football *and* at Mathematics then he is a threequarter "!

The yardstick I have taken has been school 1st XV and A-level Maths Group III or IV at Oxford or Cambridge.

The evidence I have covers five years at Fettes College, in which time seven boys got A-Level Maths *and* were in the 1st XV: of the seven, six were threequarters and the seventh was a wing forward (and also a bad mathematician who was lucky to get his Group IV).

The connection first occurred to me at Edinburgh University, where I noticed that the only four members of the Honours Maths class who played senior rugby were all threequarters.

If anyone wishes to send me figures from their experience I would be pleased to analyse them.

Yours etc., BRUCE M. MCKENZIE

Priory House, Priory Road, Stamford, Lincolnshire

To the Editor of the *Mathematical Gazette*

SIR,—The recent review of Professor Tarski's *Logic, Semantics and Metamathematics* refers to the loss which foundation studies sustained in the tragic death in a motor accident of Dr. J. Kalicki, a young colleague of Professor Tarski's. I wish to point out that Dr. Kalicki himself was the driver of the car in which the accident occurred, and not Professor Tarski, as the reviewer stated.

Princeton University Yours, etc., DANA SCOTT

QUERY

Can any reader throw light on the following definition, or suggest any polyhedra which conform to it?

Doubly reversible polyhedron. A polyhedron which exhibits, in the faces touching the base, a series repeated twice. So in a *trebly reversible polyhedron*, etc., the series is repeated thrice, etc.

Source: *Century Dictionary*, 1889.

Credit: Prof. Pierce. H.M.C.

ANTICIPATING THE WORK OF THE SIXTH FORM

By E. Œnone Wolstenholme

The main object of this address† is not so much to express my own views on the matter, as to stimulate discussion among the many experienced teachers here, which may encourage the inexperienced, or timid, teacher to loose his grim grip on the printed examination syllabus, and refresh himself and his pupils by occasional profitable digressions from the printed path.

By " anticipating the work of the Sixth Form " I am assuming that we mean providing the pupil with such mathematical training in the main school that he reaches the Sixth Form with a mind which is eager, alert, and inquisitive, and with a sufficient equipment of mathematical knowledge to enable him to explore some of the mathematical realms which are awaiting his attention.

There seem to be three main methods by which we may attempt to attain this objective :

First comes the way in which every mathematical topic is presented, from the first day the pupil enters the school, so that in the most elementary work he realises that *understanding* is his business, not the acquisition of tricks whereby an approved answer may be obtained.

Second come informal excursions into topics not usually included in the main-school syllabus ; such excursions will be made to satisfy the curiosity of the class concerning the application, or possible extension, of the work in hand.

Third comes the formal course which will include calculus, theoretical trigonometry, applied mathematics, and coordinate geometry.

Let us consider first the method of presentation. This is the most important of the three methods I have proposed, and indeed, pervades the other two, for it is in its presentation that mathematics may be transformed from a mere subject of instruction, to an inspiration to the pupil, and thus become a true vehicle of education. To inspire the pupil mathematically the subject must be presented in such a way that his mind is completely absorbed in the problem in hand ; sufficient guidance must be given to prevent the pupil from becoming frustrated by his impotence, and yet help must be withheld to such an extent that the pupil is always striving to make the next step himself—and that he makes it unaided, the teacher's task being, not to tell the pupil what to do, but to ensure that the steps attempted are within his mental capacity. It is fatally easy to teach mathematical processes in a purely authoritarian manner, so that the reasonably industrious pupil can learn how to achieve the right answer to a specified type of problem. Such teaching may produce fairly good examination results up to a certain not-very-high level, but it certainly will not produce mathematicians ; the clever pupil will be discouraged by boredom, the weak pupil will be mystified, and the mediocre pupil will painfully perform the allotted task without interest or enthusiasm. A good mathematics lesson should be as much fun as a game of hockey or an afternoon's rowing, and this can only be accomplished by planning the lesson so that the pupil is, as I have already said, always striving to discover facts for himself. To take a very elementary example let us consider the use of indices. The pupil may be told that it is usual to write $a \times a \times a \ldots n$ times as a^n ; leave it to him to discover that $a^m \times a^n = a^{m+n}$ by working from the definition with simple numeral examples, writing out $a^3 \times a^4 = (a \times a \times a) \times (a \times a \times a \times a) = a^7$ etc., until he sees the labour-saving device of adding the indices. Geometry provides many opportunities for the thrill of discovery, and also for the thrill of logical justification of an intuitive

† At a Meeting of the Exeter Branch of the Mathematical Association, April, 1957.

hunch. The love of logical exposition seems to be a more mature taste, though it can be inculcated and encouraged from very early stages, if care is taken to grade the problems so that at first the pupil is only required to make one deductive step; soon he will be required to make two deductive steps, and the number of independent deductive steps will then be increased only when, and as, the class shows that it is ready to cope with such situations.

Another way in which the presentation of the most ordinary subject-matter can help to create the future mathematician is in the setting down of examples. The insistence on clear statements and careful paragraphing make the pupil himself think clearly and acquire a nice use of words. A good device for encouraging the nice use of words is to let the pupils themselves produce the enunciation of the theorems they discover in class. A concerted effort of the whole class will often produce an astonishingly good statement, so that with very little " polishing-up " it may be written down neatly and kept for reference.

Then again, many problems are capable of several solutions—encourage the children to devise alternative solutions—teach them to be dissatisfied with clumsy algebraic expressions—teach them to be unhappy if they have not exploited all the possibilities of a geometrical diagram.

Let us now turn from the presentation of the subject-matter to the subject-matter itself. The syllabus for the main school will presumably have been devised so as to cover the topics demanded by the examining body favoured by the school, and also those other topics which the teacher himself considers essential for the good general education of the average pupil.

As he proceeds up the school, learning new facts, or acquiring new mathematical tools, the pupil should ask himself what other uses these tools can be put to, and whether further facts could not be deduced as riders to the ones he knows, or perhaps a sequence of similar facts deduced along parallel lines. If this curiosity can be stimulated by even a small glance at the vast world of Mathematics which lies beyond the main-school course, much will have been done to produce in the pupil that mathematical mind which we wish him to have.

Consider, for example, loci. Most pupils who do not make a special study of mathematics in the Sixth Form leave school with an idea that Geometry is a subject concerning straight lines, with a few circles thrown in for good measure. Why not proceed to consider the other conic sections as loci—the parabola, the ellipse, and the hyperbola? And at the same time, why not grasp the opportunity to emphasise that the graphs drawn on squared paper are merely algebraically-expressed loci whose essential character is no different from the geometrically expressed loci? And again, the duality which exists between lines and points can be exploited by a very elementary and purely practical consideration of envelopes.

Or to take another example, suppose we consider logarithms. Because of their great usefulness it is expedient to teach these as early as possible, but even though the pupils are young and mathematically immature, they will grow up with a much clearer understanding of logarithms if it is emphasised from the start that the actual base used is of relatively little importance in numerical work, and that, in fact, we do use different bases to suit our convenience. It is quite simple, and well worth the expenditure of time involved, to let the pupils construct their own log tables to some simple base, such as 2, so that when eventually the base e is required, it will not be such a shock to be expected to use a base other than 10. Moreover, the brignter children may be interested to know how to change from one base to another by the use of the identity $\log_b a \times \log_c b = \log_c a$. This again fits in very neatly with a little theoretical exploration of surds and indices.

Or yet another example could be taken by considering numbers. The way

to the later consideration of complex numbers may well be paved by the insistence, at every stage of the work, that each successive new type of number, be it natural number, fraction, negative number, or irrational number, is invented to fill a mathematical need, and which must be given its rules of manipulation so that it supplements the previous scheme, without in any way upsetting it. Here I should like to make a special plea against the iniquitous practice of telling pupils that $i = \sqrt{-1}$ and is a useful "getout" for otherwise insoluble quadratic equations. I don't suppose anyone here tonight would be so wicked, this is merely a plea to eradicate it when and if you meet it.

The conception of limits can be introduced in the discussion of tangents. Here it is wise to discuss tangents to circles in conjunction with tangents to the various other curves which may be drawn on graph paper.

Or again, the curious pupil will not be satisfied with the tedious arithmetical method of working out compound interest sums, and with a quick division one may well have time to discuss Arithmetic and Geometric Progressions, with their applications to simple and compound interest, with even a further digression to discuss Investments and Annuities.

The primary aim of the few digressions I have mentioned, and the many more I expect you have been thinking of, is to lure the mind of the pupil from the purely metrical aspects of the subject, and induce him at least to see the possibilities of a more generalised Mathematics. The prospect is at first bewildering, and frequent returns to the safety of definite and real numbers are necessary, though to the better divisions the prospect of symbolic mathematics should become more and more attractive, and the necessity to pin the work down to a numerical basis less and less urgent.

Finally let us consider the formal course with a syllabus definitely more advanced than that of the ordinary main-school course. Attending such a course gives the budding mathematician his first opportunity to work with symbols instead of numbers, and to treat the symbols as existing in their own rights, not as inferior substitutes for numbers. Providing such a course gives the teacher a real opportunity to inculcate in his pupils the mental attitude he wants his future specialists to have.

If some calculus has not been included in the main-school course, it will certainly be included in this course, and even if it *has* been included in the main-school course, there will be considerably more calculus in this special course. Calculus, I think, provides one of the best elementary means of training the pupil to think mathematically, and also perhaps, is the subject most dangerously capable of providing a mathematical tool to be used automatically and without understanding. The idea of limits will already be familiar, and so also will the conception of the gradient of a curve, so that the pupil will not take too unkindly to the idea of finding the ratio of two small quantities, and the limiting value of such a ratio. The pupil will realise the immense potentialities of this new technique and will be anxious to explore this exciting new world of differential calculus. It is well, however, to keep his speed in check, otherwise his anxiety to exploit the new method may result in his losing sight of the basic principles involved. With this in mind, I seldom permit the use of the second derivative in the discussion of maxima and minima until a very considerable amount of work has been done by considering the variations in sign of the first derivative—often not until we return to maxima and minima for a second reading. The work in Calculus and Coordinate Geometry opens the door which will lead eventually to abstract mathematics, and it is with that end in view that the early work should be planned. Nothing should be taught, or implied, or assumed which will later have to be repudiated, for intellectual honesty and integrity are the most valuable attributes of a mathematical mind.

In this formal pre-Sixth Form course I would always include Applied

Mathematics for its mathematical value, showing how mathematics can be used to discover the probable behaviour of the physical world, and also how, by applying pure mathematics to the observable behaviour of the physical world, its unobservable nature can be deduced. I would also include applied mathematics for the purely practical and non-mathematical reason that it makes the average pupil, who will one day be the average non-mathematical adult, aware of various natural phenomena which hitherto he has hardly noticed. It will also make him aware of the probable behaviour of, say, balls projected freely under gravity, of pile-drivers, of balls which are hit, or kicked. He will begin to understand tacking in sailing, he will be thrilled at the applications of his statics to find centres of gravity, resultant forces, stresses in beams, etc.

I should like to finish with a plea for the inclusion in the Mathematics course of historical references. The history of the subject can, I am sure, be made both interesting and a source of inspiration to the student. What could be more interesting than to see the study of series develop from the early Egyptian efforts to study one particular geometric series, later immortalised in the nursery jingle about the man who went to St. Ives, to the most recent study of the convergence and divergence of infinite series? Or again, surely it is a matter of interest to trace the development of geometry from the utilitarian surveying of the Egyptians, through the formalised Euclidean geometry of the Greeks, to projective and algebraic geometry and the abstract geometry of the last century? At a very early stage, when the child first learns to translate a verbal statement into a literal formula, he will be interested to hear that such work was once a matter of serious mathematical research, and he will surely be enthralled by an account of Diophantine syncopated algebra. And is no inspiration to be derived from the development of the theory of numbers, from man's first efforts to count, to the study, of the real and complex variable and the modern study of topology? And some inspiration, too, is surely to be derived from knowing a little of the men who did this work—how they co-operated and how they bickered—how much, like the work of Archimedes, was done as war-time research—how much was done under the patronage of wealthy and enlightened lovers of culture as was the work of Galileo. Time can certainly be wasted if too much is spent in the consideration of other people's efforts, but much of value will be lost if nothing is known of this historical background. I do not suggest that large chunks of your hardly-won five or six periods a week should be spent on this work, but a little class time could occasionally be spared, and the pupil's interest aroused sufficiently for him to do the rest browsing in the library.

19, *Ellers Drive, Bessacarr, Doncaster, Yorks.* E.Œ.W.

1913. Mr. F. C. Bell, assistant secretary, tried to put the subject in perspective. With the danger, he argued, went a prospect of hope. Was it realized, he wondered, that a pound of uranium 235—a mere *square* inch of the stuff—could drive a train from Euston to Glasgow and back 44 times?—*The Times*, July 8, 1955. [Per Mr. P. W. P. Browne.]

1914. I shall never forget seeing a young man at Turin, who had learnt as a child the relations of contours and surfaces by having to choose every day isoperimetric cakes among cakes of every geometrical figure. The greedy little fellow had exhausted the art of Archimedes to find which were the biggest.—J. J. Rousseau, *Emile*, p. 111. [Per Mr. L. W. H. Hull.]

CAREERS FOR WOMEN GRADUATES IN MATHEMATICS

By I. W. Busbridge

The question is often asked, " What can a woman do if she has a degree in mathematics and does not wish to teach? " Thirty years ago this was a very real problem, but today no woman mathematician need fear that she will be unable to find suitable employment. Most of the careers mentioned in this article are ones which the writer's own pupils have taken up (or given up!). It is hoped that this will be the first of a series of articles dealing with careers for boys and girls with a mathematical bent.

It must in the first place be emphasised that all careers for which the requirements are a university education, a good personality, and a helpful outlook on life, are open to the mathematician as well as to the arts graduate. One of my pupils has become a *House Property Manager*. The training for this combines practical work on a housing estate with study for an approved technical examination. The period of training usually lasts for two years, but the trainee is paid a small salary. Another pupil is a *Hospital Almoner*, for which the training is also two years. During the first year the trainee studies for a diploma in social science and her practical training comes mainly into the second year.

Mathematicians can, of course, compete for the *Administrative Branch of the Civil Service* (and other branches), but few have a sufficient knowledge of world events to stand up to the gruelling viva. The *Patents Office* takes a few mathematicians, but most of the posts of interest to mathematicians come under the *Scientific Civil Service*. For this there are two modes of recruitment. Application can be made to the Civil Service Commissioners for appointment to permanent posts, but new graduates are seldom, if ever, successful. Alternatively, application can be made direct to some Government Department or Research Establishment for appointment to a post which is " temporary " in the first instance, but which can become " established " after two or three years if the mathematician does well. Many of these posts involve interesting work and they are open to women as well as men. Several of my pupils have been at the Royal Aircraft Establishment and all have been engaged on interesting work.

For women with an interest in *Statistics* there are some good Civil Service posts. For the highest positions recruitment is, in general, by examination, but there are lesser posts of an interesting kind. One of my pupils is in the department concerned with Social Surveys. She is employed on the collection and presentation of data about subjects such as " Government hearing aids " and " Boarded-out Children ". Another has just been appointed to a statistical post in Nigeria (under the Colonial Office).

There appears to be a shortage of well-qualified statisticians. A university certificate or diploma involves from one to two years of post-graduate work, but in the end, the statistician is likely to have the choice of several well-paid posts. Alternatively, some people work for the examinations of the Royal Statistical Society whilst doing a full-time job.

In Government research establishments, industry, and the universities, more and more electronic computers are being installed. Even city councils are beginning to acquire them. *Linear Programming* provides interesting work for anyone with a mathematical bent and women are well suited for this work. If the computer is only used to work out the pay packet, high academic qualifications are unnecessary, but anyone programming for research problems should have very considerable mathematical ability.

Industry is, in general, reluctant to employ women mathematicians for anything except routine work, where women seem to be more uncomplaining

than men. However, a few large firms have begun to use women for important and interesting work, such as the performance of aircraft, linear programming, etc. Some firms manufacturing goods for domestic use employ a few women mathematicians for market research. Any training is given by the firm, and this may involve practical work in the form of door-to-door enquiries in some large city. To survive this treatment, one must believe in the value of the commodity!

Work in industry often involves very long hours. This is especially true of the printing industry, where some mathematicians are needed as readers. At least one well-known printing firm makes the readers clock-in at 8 a.m., and they do not clock-out until 5.30 p.m.

It is now possible for women to take up *Actuarial Work* and *Accountancy*, but the prospects are much less good for a woman than for a man. The training in each case is long, and that for an actuary may take five or six years after a university degree. The student is paid a salary which starts low and increases with each examination passed. Two of my pupils started on the actuarial training, but both are now in the teaching profession. One of them wrote to me that the work which she was doing " could be equally well done by a well-constructed robot ", but that her " lessons, however bad, could have been given by no one else in the world! "

Finally, what about matrimony? Some girls with a mathematical bent refuse to study mathematics beyond O-level of the G.C.E. because it spoils their chances of marriage. On the other hand, the reason advanced by many industrial firms for not employing women mathematicians is that they all leave to get married! In my own experience the industrialists are right. The teaching profession owes many recruits to matrimony and these women are helping to solve the problem of the supply of junior mathematics mistresses, but few of them are likely to take senior posts. There is now a shortage of senior mathematics mistresses. How is that problem to be solved?

St. Hugh's College, Oxford I.W.B.

1915. On the opposite wall he could see the six-inch map of the Chaddesbourne estate, its irregular equilateral triangle outlined with a wash of red water-colour. That shape, which he had always considered as permanent and immutable as the triangular outline of England itself, no longer represented the geographical truth. The equilateral triangle had become an isosceles. —F. B. Young, *This Little World*. [Per Mr. B. J. Barnes]

1916. Time $(T) = \pi\sqrt{\dfrac{\text{length }(L)}{\text{gravity }(G)}}$ (giving the answer in feet). Gravity in London is given as 32·19. Pi equals 3·14159. Although the correct value of π is 3·14159, 3·1416 or even 3·14 are used unless very close calculations are required.

If it is required to find the time taken for one vibration of a 12 in. pendulum. the calculation is :

$$3{\cdot}14159 \times \sqrt{\frac{L}{32{\cdot}19}}$$

As the square root of 32·19 is 5·67434, the equation becomes $3{\cdot}14159 \times \dfrac{L}{5{\cdot}67434}$, giving the length as 0·553 seconds. Thus, a pendulum 12 in. long performs one vibration in 0·553 secs.—*Watchmaker, Jeweller and Silversmith*, August, 1955. [Per Mr. M. A. Porter.]

WHY MATHEMATICS?

By W. G. Bickley

Most of us, I suppose, can quote a few phrases read or heard, which have greatly affected our outlook on life in general and our own work in particular. Of about half a dozen such, two came vividly into my mind as I recently attended, at the University of London Institute of Education, a course of lectures in which a closer link between the sixth form and the University courses in mathematics was sought. The first goes back to my youth, and comes from Ruskin—" Know what you have to do—and do it! " Ruskin goes on to say that insufficiency of means is far less frequently a source of failure than an inadequate understanding of the thing to be done. The second is more recent, and came as a gentle rebuke from more than one of the nurses at St. Bartholomew's Hospital, during my stay there after becoming blind, when I apologetically asked for some small non-routine service— " What are we here for? " Surely Ruskin's precept and the nurses' question are very relevant to our work.

What, then, qua teachers of mathematics, *are* we here for, and what *have* we to do? In other words, what is mathematics for? Surely the answer is not remote, or difficult to formulate. The function of mathematics is the same as that of all other worthy—and of some unworthy—fields of human endeavour: to satisfy some human needs and desires. It is not, any more than anything else—except perhaps death—an end in itself, but is a means to other ends.

Some will tell you that mathematics is the Queen of the Sciences; others that she is their handmaiden. I suggest that the latter is the more honourable function. The 3-4-5 triangle was known as a piece of technological know-how by the Egyptian surveyors and was not discovered by Pythagoras or by Euclid. Zero was invented by the mediaeval merchants. Logarithms were sought for, and trigonometry developed, to aid the navigators who roamed the world after Columbus discovered America. Newton invented the Differential and Integral Calculus as a tool to explain the cosmology to which he had been led by the work of Copernicus and Kepler. Radio waves were discovered because Hertz tried to verify the results of Maxwell's mathematics, which in its turn was based on the experimental discoveries of Faraday. More recently the work of Heaviside has led to the extensive study of the Laplace transform, and towards the Atlantic telephone. The techniques of the modern automatic computer threaten completely to revolutionise the administration of industry and commerce, and even of government itself. Mathematics, science, and technology—each nourishes, and is nourished by the other two.

What, then, are we teachers of mathematics here for, and what have we to do? Surely to show our pupils and students how this strand of mathematics is woven into the fabric of life. We must instil into them the desire to learn, and then aid their quest for knowledge.

For most of them, mathematics will be an ancillary subject, means to some extra-mathematical end. Even for those few who specialise in mathematics, and become professional mathematicians, the same will be, in the main, true. In their professional work they will, except in the rarest cases, whether individually, or as members of some team or institution, " apply " their mathematics to some other field. On the other hand, however, many will not make much use of mathematics at all beyond the fifth form. All the more need, therefore, in view of the fundamental role which mathematics plays in the foundations of our scientific and technological civilization, that these boys and girls should also understand something of what mathematics is, what it

does, and how it does it. For both groups, the teaching of mathematics should be firmly anchored to reality.

But what, some of you—if you have followed me so far—will say, what is to be done for those who will ultimately enter the inner shrine, and especially those destined to become first-class "pure" mathematicians? There is, really, little we can do for them—they do it for themselves—except create a mathematical climate in which their genius can flourish and ripen. The best thing we can do for them is to raise the general tone of mathematical teaching.

This applies at all levels. We in the universities are very dependent upon the work done in the secondary schools, and they in their turn on that done in the primary schools. May we exaggerate somewhat, and say that many who teach in the lower forms do not know enough mathematics, and many of those who teach the sixth forms do not know anything else? What seems to me most necessary is a new outlook on the subject, on the lines already indicated in the foregoing.

Mathematics is ultimately a set of abstractions. But we must not supply these abstractions ready-made. We must carefully put our pupils in the position where they can—where they must—make them for themselves. Much has been written about the improvement of mathematical teaching. But the pioneer work of such men as John Perry, W. C. Fletcher, Percy Nunn, Godfrey and Siddons—to mention a few—work done when I was unborn or during my boyhood, has not yet been properly absorbed into the philosophy or practice of our craft.

There is no doubt that—measured by the examination yard-stick—elementary geometry is by far the least successful of the subjects usually regarded as part of school mathematics. Partly this is due to the unresolved, and often subconscious, doubt in the minds of many teachers whether they are to teach space knowledge or logic. Partly, too, it is due to what one of Mr. Siddons' less able, but more perceptive, pupils discerned in the nature of the subject, that it is "a series of flukes". But what form do the majority of the exercises in most of our text-books (not only of geometry) take? "Prove this," "solve that" . . . and who cares, or is likely to care, unless they have developed that popular apotheosis of modern culture the "cross-word mind"? Puzzles, not problems, are set! One does not think of trying to "prove" anything unless one has discovered it, and has reasons—even if only intuitive—for believing it likely to be true. And it is even more fun to make puzzles than to solve them! The fun for most of us, for most of our time, is in doing, making, finding out . . ., *not* in passively seeing, receiving, and being told. Also, let us get away from the obsession with "answers", and give more attention to methods, to the strategy and tactics. It is more fun—and much more instructive—to solve a problem in three ways than to solve three problems in the same way. Even in these examination-ridden days, plagued as we are by them from 11 + to 70 –, and on the lowest level, I believe such an approach would yield handsome dividends. The criteria of a man's education are not only the questions he can answer, but even more those he asks. But more than that, I believe that if we wish also ultimately to encourage an appreciation of the aesthetic and philosophical side of mathematics, this is also the most promising way to success. It will produce not only better scientists and engineers, but also better mathematicians. We in the Universities may then get students who have some inkling of what mathematics is and does. If we can show how deeply mathematics has its roots in life, in the needs and desires of mankind, then the handmaiden may, indeed, turn out also to be a Queen!

Imperial College W.G.B.

BEGINNERS' CORNER

This corner is intended for members who are not yet frequent contributors to the *Gazette*. Regular contributors are invited to write opening sections of articles on elementary topics, and new-comers are invited to continue them in any way that seems to arise naturally.

1. Triangles whose angles are in arithmetic progression.

Let ABC be a triangle whose angles are in arithmetic progression. The "middle" angle is then $60°$; call it A.

An obvious triangle to consider is one in which

$$A = 60°, \quad B = 40°, \quad C = 80°,$$

and an obvious starting-point is to draw the bisectors AP, BQ, CR of the vertical angles, meeting in the in-centre I.

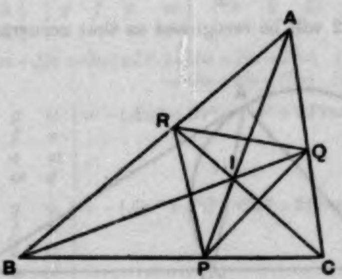

Fig. 1.

The triangle RBC is isosceles ($100°$, $40°$, $40°$), so that

$$RB = RC,$$

and the triangle BQC is isosceles ($20°$, $80°$, $80°$), so that

$$BQ = BC.$$

The angles round I are $50°$, $60°$, $70°$, which are also in arithmetic progression, and so the triangle CPI is isosceles ($40°$, $70°$, $70°$), giving

$$CP = CI.$$

Also $\angle QIC = 60° = \angle RAQ$, so that the quadrilateral $IRAQ$ is cyclic; but AI bisects the angle RAQ, so that

$$IR = IQ.$$

Hence the triangle IQR is isosceles ($120°$, $30°$, $30°$).

It follows that the angles of the triangle RAQ are $50°$, $60°$, $70°$, in arithmetic progression; that the triangles CIQ, CAR are of $40°$, $60°$, $80°$ type; and that the angles of the triangle BIR are $20°$, $60°$, $100°$, also in arithmetic progression.

E. A. Maxwell

2. Triangles whose sides are in geometric progression.

Let ABC be a triangle whose sides are

$$BC=1, \quad CA=x, \quad AB=x^2.$$

By the triangle inequalities,

$$x+x^2>1,\ x^2+1>x,\ 1+x>x^2.$$

There is no real loss of generality if we take $x<1$, so that BC is the largest side and AB the smallest. The two inequalities $x^2+1>x$ and $1+x>x^2$ are then satisfied automatically. The other is

$$x^2+x+\tfrac{1}{4}>\frac{5}{4}$$

or (each side being positive)

$$x+\tfrac{1}{2}>\frac{\sqrt{5}}{2},$$

so that

$$\frac{\sqrt{5}-1}{2}<x<1.$$

The ratio $(\sqrt{5}-1)/2$ will be recognised as that occurring in the theory of " golden section ".

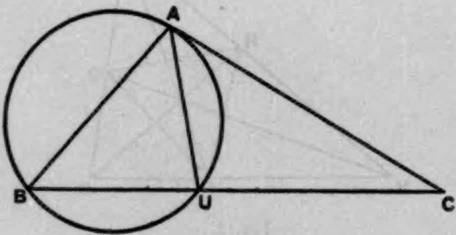

FIG. 2.

Let the circle be drawn passing through B touching AC at A and meeting BC again at U. Then the triangles ABC, UAC are similar, so that

$$UC=x^2, \quad AU=x^3.$$

In this way, a chain of triangles, of sides

$$x^n,\ x^{n+1},\quad x^{n+2}$$

may be constructed.

E.A.M.

3. A " Feuerbach " problem.

It is familiar that the nine-points circle of a given triangle is the locus of the centres of the rectangular hyperbolas through its vertices. The nine-points circle touches the inscribed circle of the triangle, and it is a matter of interest to identify the rectangular hyperbola whose centre is the point of contact. This hyperbola is, in fact, the one passing through the incentre.

Members are invited to give a *neat* proof of this result, and to discuss the general problem of associating particular points with particular rectangular hyperbolas.

E.A.M.

CLASS ROOM NOTES

14. Evaluation of a well-known determinant in Co-ordinate Geometry.

I find the following method of evaluating the determinant

$$\begin{vmatrix} a & h & g & u \\ h & b & f & v \\ g & f & c & w \\ u & v & w & d \end{vmatrix}$$

simpler and neater than the conventional methods given in standard text-books.

Putting $D=\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$, and denoting the co-factors of $a, h, g, \ldots$ in D by $A, H, G, \ldots$ respectively, we have

$$\begin{vmatrix} A & H & G & 0 \\ H & B & F & 0 \\ G & F & C & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} . \begin{vmatrix} a & h & g & u \\ h & b & f & v \\ g & f & c & w \\ u & v & w & d \end{vmatrix} = \begin{vmatrix} D & 0 & 0 & (Au+Hv+Gw) \\ 0 & D & 0 & (Hu+Bv+Fw) \\ 0 & 0 & D & (Gu+Fv+Cw) \\ u & v & w & d \end{vmatrix}$$

$$= -(Au+Hv+Gw)uD^2+(Hu+Bv+Fw).(-vD^2) \\ -(Gu+Fv+Cw).(wD^2)+dD^3$$

Hence $$D^2\begin{vmatrix} a & h & g & u \\ h & b & f & v \\ g & f & c & w \\ u & v & w & d \end{vmatrix} = -(Au^2+Bv^2+Cw^2+2Fvw+2Gwu+2Huv)\,D^2 + dD^3,$$

and so $$\begin{vmatrix} a & h & g & u \\ h & b & f & v \\ g & f & c & w \\ u & v & w & d \end{vmatrix} = -(Au^2+Bv^2+Cw^2+2Fvw+2Gwu+2Huv) + dD. \qquad \ldots\ldots(1)$$

On putting $d=0$,

$$\begin{vmatrix} a & h & g & u \\ h & b & f & v \\ g & f & c & w \\ u & v & w & 0 \end{vmatrix} = -(Au^2+Bv^2+Cw^2+2Fvw+2Gwn+2Huv).$$

Lingaraj College, Belgaum, India M. S. Trasi

15. Calculation of Logarithms.

I have always felt that it would be useful in teaching logarithms to be able to show how one could calculate logarithms to four significant figures by an elementary method not involving the exponential or logarithmic series. To show the method used by Napier is too long, but one can adapt it in the following way.

Make a table of values of a^n, where $a=1{\cdot}001$, from $n=1$ to 13, and also tables of powers of 2 and 3. One can then find products of powers of 2, 3 and 10 which lie between a^1 and a^{13}, *e.g.*

$$\frac{3^4}{2^3 . 10}=1{\cdot}0125, \quad \frac{2^{13}}{3^4 . 10^2}=\frac{8192}{8100}=1{\cdot}1011358025$$

and $$\frac{3^3 . 10^7}{2^{28}}=\frac{270000000}{268435456}=1{\cdot}005828380.$$

Linear interpolation from the table of a^n gives the logarithms of these numbers to base a, and hence we find

$$4 \log_a 3 - 3 \log_a 2 - \log_a 10 = \log_a 1{\cdot}0125$$

$$\simeq 12 + \frac{433780}{10120662} = 12{\cdot}428608. \quad \ldots\ldots\ldots\ldots\ldots(1)$$

Similarly

$$13 \log_a 2 - 4 \log_a 3 \ - 2 \log_a 10 \simeq 11{\cdot}299548, \quad \ldots\ldots\ldots\ldots\ldots(2)$$

$$3 \log_a 3 + 7 \log_a 10 - 28 \log_a 2 \simeq \ 5{\cdot}814290. \quad \ldots\ldots\ldots\ldots\ldots(3)$$

Solving (1), (2) and (3),

$$\log_a 10 \simeq 2303{\cdot}71, \ \log_a 2 \simeq 693{\cdot}486, \ \log_a 3 \simeq 1099{\cdot}15.$$

Whence $\qquad \log_{10} 2 = \log_a 2/\log_a 10 \simeq 0{\cdot}30103$

and $\qquad \log_{10} 3 \simeq 0{\cdot}47712.$

In the same way by making use of the facts that

$$\frac{3 \,.\, 7^3}{2^{10}}, \ \frac{11^2}{2^2 \,.\, 3 \,.\, 10}, \ \frac{7 \,.\, 13}{3^2 \,.\, 10}, \ \frac{17^2}{2^5 \,.\, 3^2}, \ \frac{19^2}{2^2 \,.\, 3^2 \,.\, 10}$$

all lie between a and a^{12} we can find the logarithms of 7, 11, 13, 17, 19 to base a and hence to base 10. From these, and the logarithms of 2 and 3, a table of logarithms of numbers from 1 to 10 at intervals of 0·5 can be made, and the answers are all correct to four significant figures.

If we define e by $\lim \{1 + (1/n)\}^n$, then $e \simeq (1{\cdot}001)^{1000} = b$, say. Then

$$\log_a b = 1000 \ \text{and} \ \log_a 10 \simeq 2303{\cdot}71,$$

and hence

$$\log_b 10 \simeq 2{\cdot}30371,$$

which is $\log_e 10$ to four significant figures.

If the binomial theorem is known to the pupils this can be improved on as follows.

If $c = 1{\cdot}000001$,

$$c^{1000} \simeq 1{\cdot}0010005 \text{ by the binomial theorem}$$

$$= a^{1{\cdot}00049959} \text{ by linear interpolation,}$$

and hence if $b' = c^{10^6}$

$$\log_a b' \simeq 1{\cdot}00049959 \times 10^3 = 1000{\cdot}49959,$$

whence $\log_{10} b' \simeq \dfrac{2{\cdot}30371}{1000{\cdot}50} = 2{\cdot}3026$, which is very close to $\log_e 10$.

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16. On sums of powers of the natural numbers.

The interesting short article by Miss Sheila M. Edmonds [1] and the two previous notes [2] on this topic have reminded me of the following simple proof, which occurred to me some years ago, of the well-known theorem that the sum of the cubes of the first n natural numbers is equal to the square of their sum.

Let $$u_n = \sum_{r=1}^{n} r = \tfrac{1}{2} n(n+1).$$

Then $$u_n - u_{n-1} = n,$$
and $$u_n + u_{n-1} = n^2.$$
Hence, by multiplication
$$u_n^2 - u_{n-1}^2 = n^3, \quad\text{...(1)}$$
and, consequently, by summation, it follows that
$$u_n^2 = \sum_{r=1}^{n} r^3,$$
i.e. $$\left(\sum_{r=1}^{n} r\right)^2 = \sum_{r=1}^{n} r^3.$$

Several other interesting results can be obtained in this simple way. For example, since
$$u_n^2 + u_{n-1}^2 = \tfrac{1}{2}n^2(n^2+1), \quad\text{.................................(2)}$$
it follows, by multiplying (1) and (2) and summing, that
$$u_n^4 = \tfrac{1}{2}\sum_{r=1}^{n}(r^7 + r^5),$$
i.e. $$2\left(\sum_{r=1}^{n} r\right)^4 = \sum_{r=1}^{n} r^7 + \sum_{r=1}^{n} r^5.$$

Similarly, we can easily deduce that
$$u_n^3 - u_{n-1}^3 = \tfrac{1}{4}n^4(3n^2+1),$$
and hence it follows that
$$4\left(\sum_{r=1}^{n} r\right)^3 = 3\sum_{r=1}^{n} r^5 + \sum_{r=1}^{n} r^3.$$

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1. Sheila M. Edmonds, *Math. Gaz.*, XLI (1957), 187–8.
2. A. N. Nicholson, *ibid.*, Note 2678 ; R. F. Wheeler, *ibid.*, Note 2687.

17. A note on convergence (supplement to Note 2704).

The proof that $(\log n)/n \to 0$ as $n \to \infty$ does not need any advanced knowledge as the last sentence of Note 2704 perhaps suggests. We start by proving the inequality
$$10^m > m(m+1) \quad\text{...(1)}$$
for integral m. The inequality holds for $m=0$, 1 and if it holds for $m=k \geqslant 1$ then
$$10^{k+1} = 10 \,.\, 10^k > (1+2/k) \,.\, k(k+1) = (k+1)(k+2),$$
so that (1) holds with $m=k+1$, whence by induction it holds for all m.

Given any $n \geqslant 1$, n lies between 10^m amd 10^{m+1} for some m and so
$$\frac{\log n}{n} < \frac{m+1}{10^m} < \frac{1}{m}$$
which proves that $(\log n)/n \to 0$ as $n \to \infty$.

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18. Geometrical proofs of some formulae for tangents.

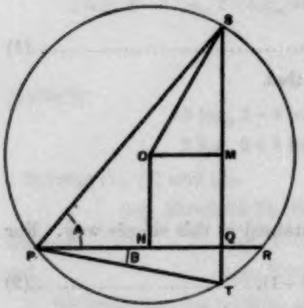

Fig. 1.

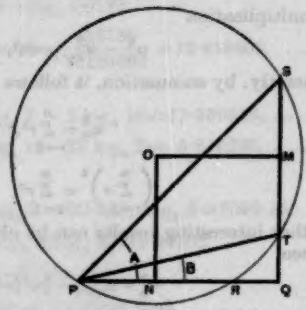

Fig. 2.

Fig. 3.

Draw PQ of unit length.
Construct right-angled triangles SPQ, TPQ (as in fig. 1) so that

$$\angle SPQ = A \text{ and } \angle TPQ = B.$$

Let O be the centre of the circumcircle of $\triangle SPT$.
Draw OM perp. to ST and ON perp. to PR.

$\therefore$ N is midpoint of PR, M is midpoint of ST and $\widehat{SOM} = (A+B)$.

$$\therefore \tan(A+B) = \frac{2SM}{2NQ} = \frac{ST}{PQ - PN + NR - QR} = \frac{TS}{PQ - QR}.$$

But $PQ \,.\, QR = SQ \,.\, QT$; $\therefore$ $QR = \tan A \tan B$.

$$\therefore \tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}.$$

In fig. 2

$$\tan(A-B) = \frac{2SM}{2NQ} = \frac{ST}{PR + 2RQ} = \frac{ST}{PQ + RQ}.$$

But $QR \,.\, QP = QT \,.\, QS$;

$$\therefore \tan(A-B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}.$$

In fig. 3

let $\widehat{POM} = x$ bisected by ON meeting $PM \perp OM$ in N.

Let $\widehat{QON} = 45°$ where Q lies on MNP produced.

$\therefore$ $PQ = OP$.

$$\therefore \tan(x/2 + \pi/4) = \frac{QM}{OM} = \frac{OP + PM}{OM} = \sec x + \tan x.$$

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19. Cricket Averages

The following problem is typical of a class of average problems which from time to time are brought to teachers of mathematics by their pupils (and even by their colleagues) with requests for explanations and often with derogatory remarks on their subject.

Two cricketers, A and B, have each taken 28 wickets for 60 runs. In the next match A takes 4 wickets for 36 runs and B takes 1 wicket for 27 runs. At first sight A has the better bowling result. But on adding the runs and wickets we find that A has taken 32 wickets for 96 runs, an average of 3, while B has taken 29 wickets for 87 runs, an average of 3 also. Thus although common sense indicates that A has bowled better than B, the mathematical average implies that they have bowled equally well.

Some thought on this problem led to a graphical illustration which gave some satisfaction to pupils and which can be used for the construction of similar problems. If the wickets and runs are plotted on the x and y axes respectively, the gradient of a line represents the bowling average. In Fig. 1 the gradient of OP represents the common average of $2\frac{1}{7}$; that of PQ, A's average of 9; and that of PR, B's average of 27. Since the points O, R, Q are collinear, the final averages of A and B are the same.

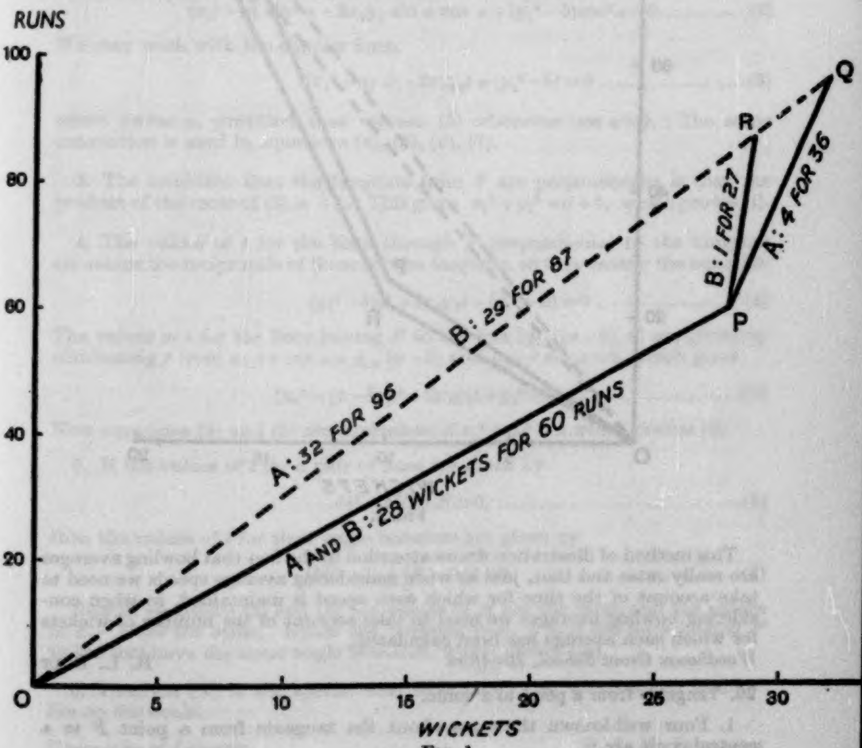

FIG. 1.

Using this graphical representation it is easy to construct a problem in which A has a higher average than B in a first group of matches and also in a second group, and yet over the combined groups A has a lower average than B. Fig. 2 illustrates such a problem. The figures obtained from this diagram are: A takes 10 for 26 (OP, average 2·6), while B takes 10 for 22 (OR, average 2·2); A takes 5 for 44 (PQ, average 8·8), while B takes 10 for 78 (RT, average 7·8); over the combined groups, A takes 15 for 70 (OQ, average $4\frac{2}{3}$) while B takes 20 for 100 (OT, average 5).

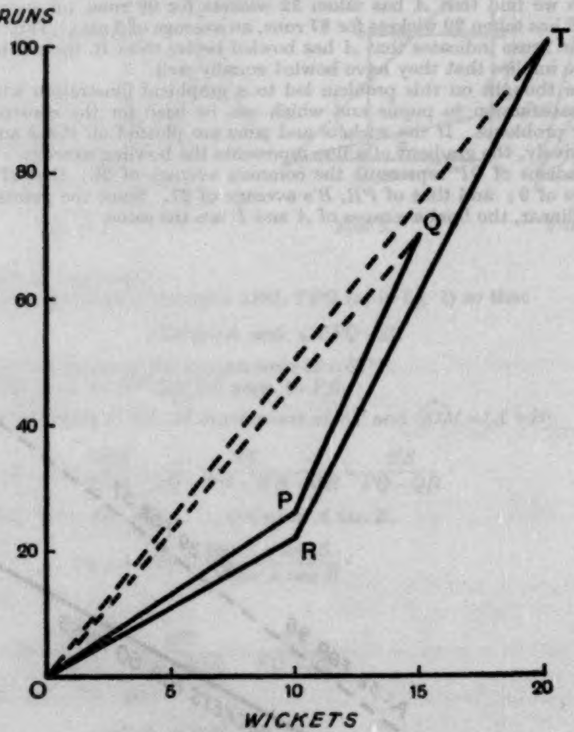

FIG. 2.

This method of illustration draws attention to the fact that bowling averages are really rates and that, just as when considering average speeds we need to take account of the time for which each speed is maintained, so when considering bowling averages we need to take account of the number of wickets for which each average has been calculated.

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20. Tangents from a point to a conic.

1. Four well-known theorems about the tangents from a point P to a central conic are:

(i) if the tangents from P are perpendicular, then P lies on a circle (the director circle); (ii) if each tangent from P is perpendicular to one of the lines joining P to the foci, then P lies on a circle (the auxiliary circle); (iii) the tangents from P make equal angles (taking no account of sense) with the lines joining P to the foci; (iv) if P lies on the conic, the tangent at P makes equal angles (taking no account of sense) with the lines joining P to the foci.

These theorems can all be proved by making use of an equation for the pair of tangents.

2. The equation of the conic, referred to its principal axes, is

$$(x^2/a) + (y^2/b) = 1 \quad \text{.................................(1)}$$

where $a > b$. Let the coordinates of P be (x_1, y_1). Any line through P may be expressed parametrically in the form $x = x_1 + r\cos\alpha$, $y = y_1 + r\sin\alpha$. The condition that the line should touch the conic is found by substituting these values for x and y in (1), and writing down the condition that the quadratic in r has equal roots. This gives

$$(x_1^2 - a)\sin^2\alpha - 2x_1y_1\sin\alpha\cos\alpha + (y_1^2 - b)\cos^2\alpha = 0 \quad \text{............(2)}$$

We may work with the simpler form

$$(x_1^2 - a)\,t^2 - 2x_1y_1t + (y_1^2 - b) = 0 \quad \text{......................(3)}$$

where $t = \tan\alpha$, provided that we use (2) whenever $\cos\alpha = 0$. The same convention is used in equations (4), (5), (6), (7).

3. The condition that the tangents from P are perpendicular is that the product of the roots of (3) is -1. This gives $x_1^2 + y_1^2 = a + b$, which proves (i).

4. The values of t for the lines through P perpendicular to the tangents are minus the reciprocals of those for the tangents, so they satisfy the equation

$$(y_1^2 - b)t^2 + 2x_1y_1t + (x_1^2 - a) = 0 \quad \text{......................(4)}$$

The values of t for the lines joining P to the foci $(\pm\sqrt{(a-b)}, 0)$ are given by eliminating r from $x_1 + r\cos\alpha = \pm\sqrt{(a-b)}$ and $y_1 + r\sin\alpha = 0$, which gives

$$[x_1^2 - (a-b)]t^2 - 2x_1y_1t + y_1^2 = 0 \quad \text{......................(5)}$$

Now equations (4) and (5) are equivalent if $x_1^2 + y_1^2 = a$ which proves (ii).

5. If the values of t for a pair of lines are given by

$$At^2 + 2Ht + B = 0, \quad \text{.................................(6)}$$

then the values of t for their angle bisectors are given by

$$H(t^2 - 1) + (B - A)t = 0. \quad \text{..............................(7)}$$

Now in equations (3) and (5) the values of H are the same and the values of $B - A$ are the same. Hence the tangents from P and the lines joining P to the foci have the same angle bisectors, which proves (iii).

6. Theorem (iv) is the special case of theorem (iii) which arises when P lies on the conic.

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MATHEMATICAL NOTES

2760. General tests for divisibility.

The Notes that have appeared on this subject in the *Mathematical Gazette* have prompted me to investigate whether there exist more powerful tests in which the number D, to be tested for divisibility by the prime number n, can be reduced successively by more than one digit at each operation.

I have come to the conclusion that the tests given in the various Notes are all special cases of more powerful and more general tests which I shall now describe and prove.

The problem is to find values of n_k such that, if the product of n_k and the last k digits of a number D is added to the remainder of D when these k digits are removed, then the resulting number will be divisible by n if and only if D is.

Now let n_k be the integer having the smallest numerical value such that $10^k n_k \equiv 1 \pmod{n}$. Then if $D = 10^k A + B$,

$$n_k(10^k A + B) \equiv A + n_k B \pmod{n},$$

so that D will be divisible by n if and only if $A + n_k B$ is. Hence, n_k has the required property.

Table 1 gives a few values of n_k and others can be calculated readily, using congruence techniques.

TABLE 1

	$k=1$	2	3	4	5	6	7	8	9	10
$n=7$	−2	−3	−1	2	3	1	−2	−3	−1	2
13	4	3	−1	−4	−3	1	4	3	−1	−4
17	−5	8	−6	−4	3	2	7	−1	5	−8
19	2	4	8	−3	−6	7	−5	9	−1	−2
23	7	3	−2	9	−6	4	5	−11	−8	−10
29	3	9	−2	−6	11	4	12	7	−8	5
31	−3	9	4	−12	5	−15	14	−11	2	−6
37	−11	10	1	−11	10	1	−11	10	1	−11
41	−4	16	18	10	1	−4	16	18	10	1
43	13	−3	4	9	−12	16	−7	−5	21	15
47	−14	8	−18	17	−3	−5	23	7	−4	9

Example. To Test 4779227210314752 for divisibility by 17.

We proceed thus :

47792272*10314752*

1st stage : $k=8$, $n_k=-1$: −10314752

37477520 (ignore final 0)

2nd stage : $k=3$, $n_k=-6$: −4512

765

3rd stage : $k=1$, $n_k=-5$: − 25

51

and hence the given number is divisible by 17.

Note that the result has been obtained after only 3 stages compared with 13 required by the test referred to in Note 2566.

The selection of suitable values of k is a matter of practice, but it is obviously advantageous to take k as close to one-half of the number of digits in the number to be reduced as is consistent with the objective of obtaining a small value of n_k.

A second table can be constructed giving values of $n_{k'}$, having the "dual" property that if the product of $n_{k'}$ and any number of leading digits of D is displaced k' places to the right and added to the remainder after these digits are removed, then the resulting number will be divisible by n if and only if D is. In this case, the fundamental congruence is $10^{k'} \equiv n_{k'} \pmod{n}$, and the proof follows the same lines as before.

For comparison with n_k corresponding values of $n_{k'}$ are shown in Table 2.

TABLE 2

	$k'=1$	2	3	4	5	6	7	8	9	10
$n=7$	3	2	−1	−3	−2	1	3	2	−1	−3
13	−3	−4	−1	3	4	1	−3	−4	−1	3
17	−7	−2	−3	4	6	−8	5	−1	7	2
19	−9	5	−7	6	3	−8	−4	−2	−1	9
23	.	8	11	−5	−4	6	−9	2	−3	−7
29	.	13	14	−5	8	−7	−12	−4	−11	6
31	.	7	8	−13	−6	2	−11	14	−15	5
37	.	−11	1	10	−11	1	10	−11	1	10
41	.	18	16	−4	1	10	18	16	−4	1
43	.	14	11	−19	−18	−8	6	17	−2	−20
47	.	6	13	−11	−16	−19	−2	−20	−12	21

Example. Employing both n_k and $n_{k'}$ techniques :

To test 2202956466814693395 for divisibilility by 31

Table 1 : $k=9$, $n_k=2$; 1629386790

3832343256

Table 2 : $k'=6$, $n_{k'}=2$; 7664

350920

Table 1 : $k=3$, $n_k=4$; 368

403 which is divisible by 31 by inspection.
(3 stages).

There are no doubt many short cuts that can be used in special cases, and as an example of one type, I shall, in testing for divisibility by 37, reduce a number of 28 digits to one of four digits in two stages (instead of the four required by the method already described).

The trick is to split D into approximately *three* equal segments, and to apply the $n_{k'}$, and n_k techniques simultaneously to the first and third segments respectively.

Thus $D=1228938081399541165888357657$

$k'=9$, $n_{k'}=1$; 1228938081

$k=9$, $n_k=1$; 888357657

2516836903

$k'=3$, $n_{k'}=1$; 2516

$k=3$, $n_k=1$; 903

4255

The method is especially powerful in this example since all the multipliers are unity, but the method can be applied even if they are not ; if any are negative, the method becomes rather tricky and is not recommended.

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2761. On Note 2592.

The "four fundamentally different solutions" to a knight's tour problem given in Note 2592 are, in fact, not fundamentally different. The fourth is merely the first reflected in a vertical or horizontal median, and the third may be similarly obtained from the second. This is not obvious from the layout of the nine digits, since the reflection gives a different part of the tour; but a tracing of the complete 36 moves immediately shows the connection.

Warnsdorff's Rule, which at best is of dubious utility, has here proved worse than useless, leading not only to this duplication of results, but also to the failure to discover the *three* other solutions which exist.

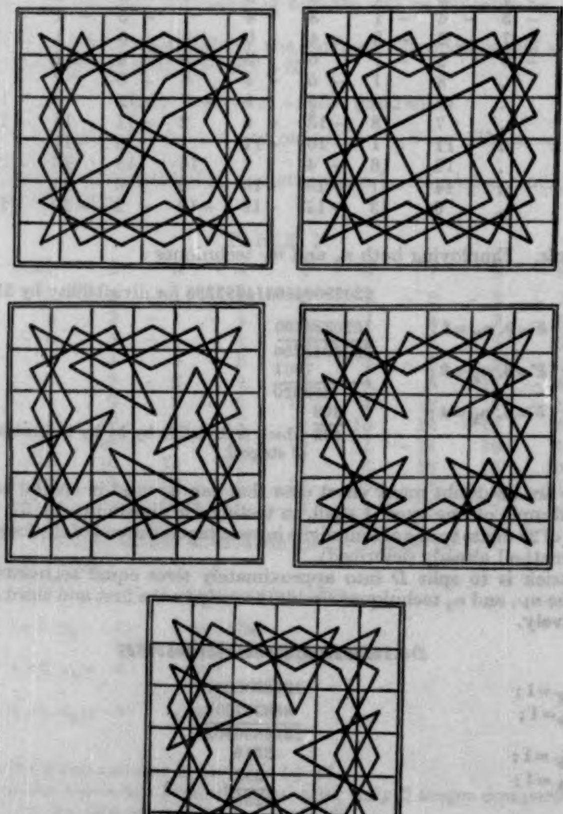

FIG. 1–5.

The five solutions were shown in my article of Dec. 1940 (*Gazette*, Vol. XXIV, page 315), and are diagrammed in all their elegance herewith. The four results of Note 2592 are shown respectively by Fig. 1, Fig. 2, Fig. 2 reflected, and Fig. 1 reflected. Figs. 3, 4 and 5 are quite distinct, and these five are the only possible solutions.

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2762. Divisibility by 47.

Mr. Langford (Note 2624) states that there is not a "pretty reasonable" test for divisibility by 47. The following is an attempt to remedy the lack of such a test.

Example. To test 992354898.

The left-hand digit is removed, multiplied by 6 and added to the number formed by the next two digits, numbers being reduced modulo 47 where convenient.

We have	992354898	
	5354898	(*A*)
	184898	(*B*)
	43898	
	1598	
	188	
	94	

(*A*) $6\times 9=54\equiv 7$

$7+92=99\equiv 5$

(*B*) $6\times 5=30$

$30+35=65\equiv 18$, etc.

All the lines are congruent modulo 47 so as 94 is divisible by 47, the original number is also. The process, of course, depends on the fact that $100\equiv 6$ (mod 47).

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Editorial Note. The same test for divisibility was sent in by Mr. J. A. K. Kashangaki and Mr. D. H. Halfpenny. Cf. note 2760 above.

2763. On the evaluation of log-sine integrals.

Introduction.

The values of $\int_0^{\frac{1}{2}\pi}\log(\sin\theta)\,d\theta$ and $\int_0^{\frac{1}{2}\pi}\log^2(\sin\theta)\,d\theta$ are both well known—the former in particular is often used at an early stage in teaching the calculus as an example of the evaluation of a definite integral when the indefinite form is unknown. Higher powers of the logarithm do not appear to have been treated in the literature. We will show here how it is possible to evaluate the general expression of integral order: more specifically we shall obtain a recurrence relationship for $Ls_n(\pi)=-\int_0^{\pi}\log^{n-1}(2\sin\tfrac{1}{2}\theta)\,d\theta$, a form from which the above integrals may be derived.

1. Let $$I=\int_0^{\pi}e^{x\log(2\sin\frac{1}{2}\theta)}\,d\theta$$

$$= \sum_0^\infty \int_0^\pi \frac{x^n}{n!} \log^n (2 \sin \tfrac{1}{2}\theta)\, d\theta$$

$$= - \sum_0^\infty \frac{x^n}{n!} Ls_{n+1}(\pi). \quad \text{.........................(1)}$$

An alternative expression for I is simply

$$\int_0^\pi e^{\log (2 \sin \frac{1}{2}\theta)^x}\, d\theta = \int_0^\pi 2^x \sin^x (\tfrac{1}{2}\theta)\, d\theta,$$

and this may be evaluated in terms of the gamma function to give

$$I = 2^x \sqrt{\pi}\, \Gamma(\tfrac{1}{2} + \tfrac{1}{2}x)/\Gamma(1 + \tfrac{1}{2}x). \quad \text{.............................(2)}$$

The duplication formula for the gamma function can be written

$$2^{x-1} \Gamma(\tfrac{1}{2}x)\, \Gamma(\tfrac{1}{2} + \tfrac{1}{2}x) = \sqrt{\pi}\, \Gamma(x),$$

and if this be used to eliminate $\Gamma(\frac{1}{2} + \frac{1}{2}x)$ in (2) we find

$$I = \pi \Gamma(1 + x)/[\Gamma(1 + \tfrac{1}{2}x)]^2. \quad \text{..............................(3)}$$

Let $D^n \equiv \left(\dfrac{d}{dx}\right)^n$ and let D_0^n denote the result of putting $x = 0$ after the differentiation.

Differentiating (1) n times and taking $x = 0$ gives

$$Ls_{n+1}(\pi) = -D_0^n I$$

$$= -\pi D_0^n \Gamma(1 + x)/[\Gamma(1 + \tfrac{1}{2}x)]^2 \quad \text{......................(4)}$$

from (3).

2. We need the derivatives of the gamma function. It is best here to work with the logarithmic form. Let $y = \Gamma(1 + x)/[\Gamma(1 + \frac{1}{2}x)]^2$ so that

$$\log y = \log \Gamma(1 + x) - 2 \log \Gamma(1 + \tfrac{1}{2}x),$$

$$D^n \log y = D^n \log \Gamma(1 + x) - 2D^n \log \Gamma(1 + \tfrac{1}{2}x). \quad \text{...............(5)}$$

Now $\left(\dfrac{d}{dz}\right)^n \log \Gamma(z) = (-1)^n (n-1)! \sum\limits_0^\infty (z + r)^{-n}$ from the expression for $\Psi(z)$, the logarithmic derivative of $\Gamma(z)$. Inserting this in (5), with $z = 1 + x$ and $1 + \frac{1}{2}x$, and proceeding to the limit $x = 0$, $z = 1$, gives

$$D_0^n \log y = (-1)^n (n-1)! (1 - 2^{1-n}) \zeta(n).$$

Hence, from Maclaurin's expansion

$$\log y = \sum_1^\infty \frac{x^n}{n!} (-1)^n (n-1)! (1 - 2^{1-n}) \zeta(n)$$

$$= \sum_2^\infty \frac{x^n}{n} (-1)^n (1 - 2^{1-n})\, \zeta(n), \quad \text{...........................(6)}$$

since the $n = 1$ term vanishes.

Accordingly $y = e^{\log y} = \exp \left\{ \sum\limits_2^\infty \dfrac{x^n}{n} (-1)^n (1 - 2^{1-n}) \zeta(n) \right\}$ and from (4) we get $Ls_{n+1}(\pi) = -\pi D_0^n e^{f(x)}$ where $f(x)$ is the function appearing in (6).

3. Now there is no general expansion for $e^{f(x)}$, though when the coefficients of $f(x)$ are suitably related particular cases may arise. In the present instance the coefficients of $f(x)$ involve $\zeta(n)$, and although the even-order coefficients

are connected via π^n and B_n there is no comparable relation for the odd orders. Accordingly no simple expression for $e^{f(x)}$ can be sought, and other methods must be found to evaluate the coefficients.

Let $y_m = D^m e^f$. Then $y_{m+1} = D^{m+1}e^f = D^m(De^f) = D^m(f'e^f)$.

Hence, using Leibnitz's theorem,

$$\begin{aligned} y_{m+1} &= f^{(m+1)}e^f + {}^mC_1 f^{(m)}De^f + {}^mC_2 f^{(m-1)}D^2e^f + \dots + {}^mC_1 f'' D^{m-1}e^f + f'D^m e^f \\ &= f^{(m+1)}e^f + {}^mC_1 f^{(m)}y_1 + {}^mC_2 f^{(m-1)}y_2 + \dots + {}^mC_1 f'' y_{m-1} + f'y_m. \quad \dots\dots\dots(7) \end{aligned}$$

In this equation take $x=0$ so that $(y_m)_{x=0} = -\frac{1}{\pi} Ls_{m+1}(\pi)$.

$$Ls_{m+2}(\pi) = -\pi f_0^{(m+1)} + {}^mC_1 f_0^{(m)} Ls_2(\pi) + \dots + {}^mC_1 f_0'' Ls_m(\pi) + f_0' Ls_{m+1}(\pi).$$

Now $Ls_2(\pi)=0$, $f_0'=0$ and $f_0^{(m)} = (-1)^m(m-1)!(1-2^{1-m})\zeta(m)$. Hence

$$\begin{aligned} Ls_{m+2}(\pi) = (-1)^m m! [\pi(1-2^{-m})\zeta(m+1) - (1-2^{2-m})\zeta(m-1)Ls_3(\pi)/2! + \\ + (1-2^{3-m})\zeta(m-2)Ls_4(\pi)/3! \dots + (-1)^m(1-\tfrac{1}{2})\zeta(2)Ls_m(\pi)/(m-1)!]. \quad (8) \end{aligned}$$

4. Equation (8) is a recurrence relationship from which successive values may be calculated. The first few are

$$\left.\begin{aligned} Ls_2(\pi) &= -\int_0^\pi \log(2\sin\tfrac{1}{2}\theta)\,d\theta = 0 \\ Ls_3(\pi) &= -\int_0^\pi \log^2(2\sin\tfrac{1}{2}\theta)\,d\theta = -\pi^3/12 \\ Ls_4(\pi) &= -\int_0^\pi \log^3(2\sin\tfrac{1}{2}\theta)\,d\theta = \frac{3\pi}{2}\zeta(3) \\ Ls_5(\pi) &= -\int_0^\pi \log^4(2\sin\tfrac{1}{2}\theta)\,d\theta = -19\pi^5/240 \\ Ls_6(\pi) &= -\int_0^\pi \log^5(2\sin\tfrac{1}{2}\theta)\,d\theta = \frac{45\pi}{2}\zeta(5) + \frac{5\pi^3}{4}\zeta(3) \\ Ls_7(\pi) &= -\int_0^\pi \log^6(2\sin\tfrac{1}{2}\theta)\,d\theta = -\left[\frac{45\pi}{2}\zeta^2(3) + 275\pi^7/1344\right] \end{aligned}\right\} \dots\dots\dots(9)$$

Beyond this the results become rather complicated, and the restriction of the expression for the odd orders to powers of π holds only up to the fifth—beyond that the odd-order ζ-functions also contribute.

5. A similar though rather simpler analysis may be used to derive

$$\int_0^\pi \log^n(\tan\tfrac{1}{2}\theta)\,d\theta,$$

and the more general form

$$\int_0^\pi \log^n[(2\sin\tfrac{1}{2}\theta)(2\cos\tfrac{1}{2}\theta)^p]\,d\theta$$

is also amenable to the same treatment. By expanding the logarithm in the form

$$\int_0^\pi \sum_{r=0}^{n} {}^nC_r \log^{n-r} (2 \sin \tfrac{1}{2}\theta) \,.\, P^r \log^r (2 \cos \tfrac{1}{2}\theta)\, d\theta$$

and comparing coefficients of P^r it is possible to deduce the value of

$$\int_0^\pi \log^{n-r} (2 \sin \tfrac{1}{2}\theta) \log^r (2 \cos \tfrac{1}{2}\theta)\, d\theta.$$

The analysis for this case is the same as in Section 1, but the function on the right of equation (4) is replaced by

$$\frac{\Gamma(1+x)\,\Gamma(1+xP)}{\Gamma(1+\tfrac{1}{2}x)\,\Gamma(1+\tfrac{1}{2}xP)\,\Gamma(1+\tfrac{1}{2}x\overline{1+P})}. \quad \text{.....................(10)}$$

The function $f(x)$ which appears in equation (6) is accordingly modified to

$$f(x) = \sum_2^\infty \frac{x^n}{n}(-1)^n[1+P^n-(\tfrac{1}{2})^n-(\tfrac{1}{2}P)^n-(\tfrac{1}{2}\overline{1+P})^n]\zeta(n) \text{..........(11)}$$

As an example, from the coefficient of x^3 in the expansion of $e^{f(x)}$ we find

$$\int_0^\pi \log^3 [(2 \sin \tfrac{1}{2}\theta)(2 \cos \tfrac{1}{2}\theta)^P]\, d\theta = -2\pi\zeta(3)[\tfrac{3}{4}(1+P^3)-\tfrac{3}{8}(P+P^2)]. \text{....(12)}$$

On expanding the logarithm and comparing coefficients of P we get

$$\int_0^\pi \log^2 (2 \sin \tfrac{1}{2}\theta) \log (2 \cos \tfrac{1}{2}\theta)\, d\theta = \frac{\pi}{4}\zeta(3). \text{ ....(13)}$$

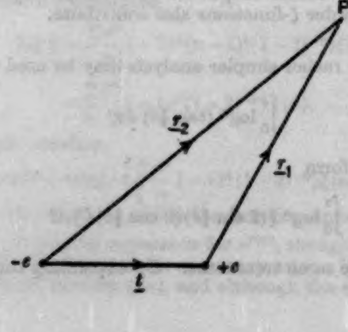
Enfield Laboratories. L. LEWIN

2754. Dipoles.

The following method for deriving the electrostatic potential and intensity due to a dipole appears to have been omitted from the standard texts, but in view of its directness of approach, it is thought that it can hardly be novel.

The method is based upon the fact that the number $(x^2 - y^2)$ can be expressed as the scalar product $(\mathbf{x}-\mathbf{y}).(\mathbf{x}+\mathbf{y})$, where $\mathbf{x}$ and $\mathbf{y}$ are any two vectors of magnitudes x and y respectively.

Potential.



In an obvious notation,

$$V = e\left\{\frac{1}{r_1} - \frac{1}{r_2}\right\}$$

$$= \frac{e(r_2 - r_1)(r_2 + r_1)}{r_1 r_2 (r_2 + r_1)}$$

$$= \frac{e(\mathbf{r}_2 - \mathbf{r}_1)\cdot(\mathbf{r}_2 + \mathbf{r}_1)}{r_1 r_2 (r_2 + r_1)}$$

Since $\mathbf{r}_2 - \mathbf{r}_1 = \mathbf{t}$, then in the limit when $e\mathbf{t} \to \mathbf{m}$ and $\mathbf{r}_1, \mathbf{r}_2 \to \mathbf{r}$, $V = \dfrac{\mathbf{m}\cdot\mathbf{r}}{r^3}$.

Intensity.

$$\mathbf{E} = e\left\{\frac{\mathbf{r}_1}{r_1^3} - \frac{\mathbf{r}_2}{r_2^3}\right\}$$

$$= e\left\{\frac{r_2^3\mathbf{r}_1 - r_1^3\mathbf{r}_2}{r_1^3 r_2^3}\right\}$$

$$= e\left\{\frac{(r_2^3 - r_1^3)\mathbf{r}_1}{r_1^3 r_2^3} - \frac{r_1^3\mathbf{t}}{r_1^3 r_2^3}\right\} \quad \text{since } \mathbf{r}_2 = \mathbf{t} + \mathbf{r}_1$$

$$= \frac{e[(\mathbf{r}_2 - \mathbf{r}_1)\cdot(\mathbf{r}_2 + \mathbf{r}_1)](r_2^2 + r_2 r_1 + r_1^2)}{r_1^3 r_2^3 (r_1 + r_2)} - \frac{e\mathbf{t}}{r_2^3},$$

and thus in the limit

$$\mathbf{E} = \frac{3(\mathbf{m}\cdot\mathbf{r})\mathbf{r}}{r^5} - \frac{\mathbf{m}}{r^3}.$$

Mutual potential energy of two dipoles.

Using the above expression for potential, together with the formula $W = \frac{1}{2}\,\Sigma\, eV$, it is clear that this method can be used to obtain the result

$$W = \frac{\mathbf{m}'\cdot\mathbf{m}}{r^3} - \frac{3(\mathbf{m}'\cdot\mathbf{r})(\mathbf{m}\cdot\mathbf{r})}{r^5}$$

for the mutual potential energy of two dipoles directly, although it is more easily obtained by using the expression for intensity together with the formula $W = -\mathbf{m}'\cdot\mathbf{E}$.

18 *Cotsford Avenue, New Malden.* B. J. PHILLIPS

2765. Motion of a top: a criterion for loop motion.

The motion of a symmetrical top is discussed in many textbooks on dynamics, usually with diagrams illustrating the motion of the axis (*e.g.*, Lamb [1], Synge and Griffith [2], Rutherford [3]). For the standard case, where a point on the axis of symmetry is fixed and friction is neglected, the axis moves between two limiting inclinations to the vertical and its azimuthal angle, ϕ, can either

(I) increase (or decrease) monotonically with time (direct motion),

or (II) fluctuate with time so that the axis makes loops in space (loop motion).

The transition case between (I) and (II) is cuspidal motion (Ia), when $\dot{\phi}$ becomes zero at regular intervals but does not change sign. The present note provides a criterion for the possibility of loop motion when θ, the inclination of the axis to the vertical, varies between given limits θ_1 and θ_2.

If A, A, C are the principal moments of inertia, M the mass and h the distance from the fixed point to the centroid of the top, then the relevant equations are

$$\tfrac{1}{2}A(\dot\theta^2 + \dot\phi^2 \sin^2\theta) + \tfrac{1}{2}Cn^2 + Mgh\cos\theta = E, \quad \text{.....(1)}$$

$$A\dot\phi\sin^2\theta + Cn\cos\theta = H, \quad \text{.....(2)}$$

where E and H are constants and n, the spin, is also constant (cf. Rutherford [3]). The three constants n, E and H fix the essential features of the motion. We shall see that the alternative set n, θ_1 and θ_2 is less restrictive, in that two values of H are possible, in general, when n, θ_1 and θ_2 are specified. In the general case, we can take $0<\theta_2<\theta_1<\pi$. The special cases where $\theta_1=\theta_2$ (precession) or where $\theta_2=0$ or $\theta_1=\pi$ can easily be discussed separately and will be ignored in the subsequent discussion. If we write $H=Cnx$ and $z=\cos\theta$, with $z_1=\cos\theta_1$, $z_2=\cos\theta_2$, then the restrictions on θ imply that $-1<z_1<z_2<1$ and $z_1\leqslant z\leqslant z_2$ during the motion. From equation (2),

$$\dot\phi = (H - Cn\cos\theta)/A\sin^2\theta = (Cn/A)(x-z)/(1-z^2), \quad \text{.....(3)}$$

and thus ϕ is monotonic with time unless $z_1<x<z_2$. .....(4)
Eliminating $\dot\phi$ between (1) and (2) gives

$$A^2\dot\theta^2\sin^2\theta = A^2\dot z^2 = 2A(1-z^2)(E - \tfrac{1}{2}Cn^2 - Mghz) - C^2n^2(x-z)^2,$$

and since $\dot z=0$ for $z=z_1$ and $z=z_2$, we get two equations involving E. From these two equations, we can eliminate E and get

$$f(x) \equiv (z_1+z_2)(1+x^2) - 2x(1+z_1z_2) + s = 0, \quad \text{.....(5)}$$

where

$$s = (2MghA/C^2n^2)\sin^2\theta_1\sin^2\theta_2. \quad \text{.....(6)}$$

Thus s is positive, with $s\to 0$ as $n\to\infty$. For a given value of n, equation (5) gives two values of x, and hence of H, for which motion with $\theta_2\leqslant\theta\leqslant\theta_1$ is possible. We should add the proviso that x must be real. From equation (5) the condition for this is that

$$(z_1+z_2)(z_1+z_2+s) \leqslant (1+z_1z_2)^2,$$

i.e., $$s(z_1+z_2) \leqslant (1-z_1^2)(1-z_2^2),$$

or $$C^2n^2 \geqslant 2MghA(\cos\theta_1+\cos\theta_2). \quad \text{.....(7)}$$

If the spin is too small, motion between the given limiting inclinations is impossible, although this restriction only comes into effect when $z_1+z_2\geqslant 0$. The inequality (7) is evidently a more general form of the well-known condition for precessional motion to be possible.

To find whether the values of H give direct motion or loop motion of the axis, we have to examine the roots of equation (5) to see if either root satisfies criterion (4). To do this, we consider separately the three cases

(a) $z_1+z_2>0$, (b) $z_1+z_2=0$, (c) $z_1+z_2<0$.

Case (a). For $z_1+z_2>0$, the graph of $f(x)$ is a parabola with a minimum at

$$x_0 = \frac{1+z_1z_2}{z_1+z_2} = 1 + \frac{(1-z_1)(1-z_2)}{z_1+z_2} > 1.$$

Thus if real solutions of equation (5) exist, one solution is $x_2>x_0>1$, which gives direct motion. To fix the position of the other solution, say x_1, note that

$$f(z_1) = (z_2-z_1)(1-z_1^2) + s > 0,$$
$$f(z_2) = -(z_2-z_1)(1-z_2^2) + s.$$

Hence $x_1 > z_1$ and loop motion can only occur if $f(z_2) < 0$, *i.e.*, if

$$s < s_2 = (z_2 - z_1)(1 - z_2^2),$$

or

$$C^2n^2 > (2MghA \sin^2 \theta_1)/(\cos \theta_2 - \cos \theta_1). \qquad \text{.......................(8)}$$

If $s = s_2$, then $f(z_2) = 0$ and cuspidal motion can occur, with the cusps at $z = z_2$. The fact that $f(z_1)$ is always non-zero confirms the standard result that the cusps cannot occur at the lower limiting inclination $\theta = \theta_1$.

The motion of the axis will be direct if $f(z_2) > 0$, *i.e.*, if

$$s_2 \leqslant s \leqslant s_1 = (1 - z_1^2)(1 - z_2^2)/(z_1 + z_2). \qquad \text{.......................(9)}$$

The second part of the inequality arises from condition (7) and ensures that the spin is large enough to make the motion possible.

Case (b). When $z_1 + z_2 = 0$, equation (5) gives $x = \frac{1}{2}s/(1 + z_1z_2) = \frac{1}{2}s/(1 - z_1^2)$. In this case the motion is uniquely defined by n, θ_1 and θ_2 and the condition for loop motion is that

$$x = \tfrac{1}{2}s/(1 - z_1^2) < z_2,$$

i.e., that

$$C^2n^2 > (MghA \sin^2 \theta_2)/\cos \theta_2, \qquad \text{.......................(8)}'$$

in agreement with condition (8), since $\sin \theta_1 = \sin \theta_2$ in this case.

Case (c). For $z_1 + z_2 < 0$, the graph of $f(x)$ is again a parabola. It has a maximum at

$$x_0 = \frac{1 + z_1z_2}{z_1 + z_2} = -1 + \frac{(1 + z_1)(1 + z_2)}{z_1 + z_2} < -1.$$

Thus one root of equation (5) is $x_1 < x_0 < -1$ and this gives direct motion of the axis. The other root corresponds to loop motion if $f(z_2) < 0$, which again gives condition (8) as the criterion for loop motion.

To summarise, motion of the axis of the top between the limiting inclinations $\theta = \theta_1$ and $\theta = \theta_2$ is possible provided the spin is large enough to satisfy condition (7). As the spin increases beyond the minimum value, there is a stage where only direct motion is possible, but once condition (8) is satisfied loop motion can also occur. Apart from the exceptional case where $z_1 + z_2$ is zero, condition (8) is not sufficient in itself to ensure loop motion. Two values of H are possible and the numerically smaller of these gives loop motion.

University of Melbourne. A. Brown

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1. H. Lamb, *Higher Mechanics*, pp. 133–9, Cambridge, 1943.

2. J. L. Synge and B. A. Griffith, *Principles of Mechanics*, pp. 432–8, McGraw-Hill, 1949.

3. D. E. Rutherford, *Classical Mechanics*, pp. 153–5, Oliver & Boyd, 1951.

1917. Newton discovered the law of gravitation; but gravitation alone would soon reduce the universe to a motionless mass; he was compelled to add a projectile force to account for the elliptical course of the celestial bodies. —J. J. Rousseau, *Emile*, p. 235. [Per Mr. L. W. H. Hull.]

REVIEWS

Modern School Mathematics. By E. J. James. (Oxford University Press)
Book II. With Answers. 7s. 6d. Pp. 205. 1955.
Book III. With Answers. 8s. 6d. Pp. 198. 1956.
Book IV. With Answers. 8s. 6d. Pp. 160. 1956.

The first book of this new series for the Secondary School was reviewed in the *Mathematical Gazette* for October 1956, p. 238. The freshness of approach to mathematics for which this book is notable is maintained throughout the three subsequent books. Not only is the mathematics carried by a series of practical topics of a worthwhile nature, such as " Farm Areas, " Money in the Bank ", and " Planning a Holiday on the Norfolk Broads ", to mention one from each book, but the mathematical content of each book is very carefully selected and arranged.

It has been the custom in most of the textbooks of the past to introduce a new mathematical topic whole with very little regard for the " mental digestions " of the pupils. Able teachers have always been aware of this and have interpolated extra stages as they were needed, leading up to the introduction of Algebra, for instance, by occasional questions and exercises designed to prepare the minds of their pupils for the new idea. Mr. James follows this plan in his books, making three or more approaches to a new concept before giving it precision by the use of the technical terms or introducing a technique of handling it. This approach seems more suitable to the child who is not particularly gifted mathematically. It is fascinating to look through the exercises in Book II, for example, to recognise the simple exercises in puzzle form which lead in Book III to formulation and the idea of generalised number; to discover that the simple slide rule for performing addition and subtraction has by Book IV become a logarithmic slide rule. Ratio is introduced and established in Book II so that it is ready as an idea and as a technique by the time it is needed in Book III for the development of the idea of the tangent of an angle.

These and other mathematical threads run up through the series of four books, sometimes carried by a practical topic, sometimes treated entirely mathematically, but always with stress laid on the need for the pupil to show his understanding by his ability to talk or write about what he is doing, and opportunities are provided for him to show initiative in the solution of problems.

As in the first book additional practice exercises are given at the end of each book, and with the answers are given some useful comments on the purpose of the exercises and suggestions on their administration. Teachers are encouraged to use their own local sources of information to introduce topics similar to those in the books and advice is given on material to use.

K. Sowden

Fundamental Mathematics. By Thomas L. Wade and Howard E. Taylor. Pp. xiv, 380. 35/6. 1956. (McGraw-Hill)

This book provides a course in elementary mathematics which has been developed over the past seven years in response to a particular situation at the Florida State University. It is intended for students whose mathematical attainment is so limited as to preclude their immediate entry into the usual first year courses in algebra, trigonometry and coordinate geometry; and for students who require a basic knowledge of mathematics for use in their other studies or for the purposes of general education, yet are not attracted to the traditional college courses. Specimen courses are drawn up to illustrate how the diverse needs of these widely differing classes of students may be met.

Mathematics is introduced as a language for the interchange of ideas, " a language of size and order, of quantities and relations among quantities ", and its usefulness as a tool is indicated. The subject is described also as " the study of number and symbols apart from the physical objects which they may represent ". Over one quarter of the book is devoted to the underlying principles and techniques of counting, number systems and the fundamental operations with rational numbers. The use of specific numbers leads to the introduction of algebraic symbols and algebra emerges gradually and spirally as arithmetical ideas and processes are generalised. Arithmetic and algebra are taken to a standard approximating to that of G.C.E. ordinary level and the beginnings of trigonometry, coordinate geometry and statistics are introduced. The work in trigonometry, for example, does not go beyond the definition of the three ratios and their application in easy numerical problems. Similar triangles, scale drawing, Pythagoras' theorem, areas and volumes are the geometrical items included. Many exercises are provided.

It is well-known that the principles underlying directed number are difficult for the student to understand. The treatment here does not distinguish between the natural numbers and the positive integers. Thus (page 7), " The symbol for the number of objects in a set of distinct objects is called *a positive integer*. The relevancy of the adjective " positive " may not become apparent until the negative integers are considered." and " It is true that in this book we shall usually represent the positive integers by the Hindu-Arabic symbols 1, 2, 3, 4, 5, 6, 7, 8, 9, ... ". The definition of negative numbers appears on page 49 as " Corresponding to each positive integer a we invent a new number $-a$ which we call minus or negative a ".

It is probably more usual in this work to take the natural numbers as the starting point. Since subtraction is not always possible with these, we invent the notion of the negative integer. This in turn gives rise to the idea of the positive integer. The positive and negative integers together constitute a new class of numbers separate and distinct from the natural (signless) numbers. The sense in which the positive and negative integers may be termed numbers needs to be discussed. The rules of operation are so chosen that these new " numbers " behave consistently with the natural numbers so that these and the positive integers may be taken to be identical for all practical purposes. The plus and minus signs, hitherto used as signs of operation, are now used in a new sense with the help of brackets to denote directed numbers. This approach to directed numbers avoids the inconsistent use of brackets which appears on pages 47–52, for instance,

P. 47. " $(+5)+(+3)=+8$ " and " $(-5)+(-3)=(-8)$."

P. 49. " $(+3)-(+8)$ is an integer c such that $(+8)+c=(+3)$ or $8+c=3$. Here $c=-5$ since $(+8)+(-5)=(+3)$."
" Here $c=(-3)$ since . . ."

P. 52. " $(+a)(-b)=-(ab)$; $(-a)(+b)=-(ab)$ ... $(-a)(-b)=+(ab)$; $(+a)(+b)=+(ab)$."

Paragraph 16.1 in a final chapter entitled " Additional Topics in Algebra " deals with " Complex Fractions " defined as being those fractions which have one or more fractions in the numerator, or denominator, or in both. The word " complex " appears with a different connotation later in the same chapter when complex numbers are discussed. The contexts are very different and consequently the danger of confusion is slight, but nevertheless it is advisable in teaching mathematics to avoid any possibility of ambiguity and lack of precision.

The beginner must find the American method of recording logarithms easier

to understand than our system of bar-notation which, however, has the advantage of being more elegant.

The aim of the authors in stressing teaching for techniques and teaching for understanding will meet with general approval. Care and thought have evidently gone into the planning of the book, which is beautifully produced and holds the attention of the reader. Superficiality seems to have been avoided, though the treatment of some topics is inevitably brief. The work on similar triangles, for example, does not include explicit discussion, or even mention, of the general idea of similarity. The book has merit and no doubt admirably fulfils the purpose for which it is written. It is doubtful whether any comparable situation exists in this country where this particular treatment and selection of topics would be appropriate, and it is this thought which has most exercised the mind of the reviewer. W. FLEMMING

The Four Rules of Number. By K. A. HESSE. Pp. 70. 2s. 6d. Teacher's Ed. Pp. 78. 6s. 1956. (Longmans)

This book is attractively produced and the material well graded. It contains examples varying from the easiest addition and subtraction, *e.g.* $1+1$ and $3-2$ to a difficult long division such as $40076 \div 84$. Diagnostic pages are included to enable particular weaknesses to be discovered. Occasionally pages with " sums with words " are inserted, *e.g.* $16-8$ is replaced by " Subtract 8 from 16 ", and so on. The whole emphasis, however, is on mechanical work in order to achieve speed and accuracy of computation.

While it is agreed that children should have frequent practice in the use of basic arithmetical bonds such as $8+9$ and 6×7, there is much less justification for applying drill methods (as this book tends to suggest) to work with larger numbers. It should also be pointed out that work in the four rules should sometimes proceed simultaneously particularly in the early stages. It is doubtful, therefore, whether examples in one rule should be concentrated on a page or adjacent pages.

One must also question whether a textbook should divorce mechanical work from suitable applications of the four rules of number to realistic situations, as teachers may be encouraged to neglect wider aims in the teaching of Arithmetic and Mathematics. Nevertheless, if this book is used, as suggested by the author, in conjunction with other approaches, the carefully graded material will be of appreciable value to the busy teacher. S. SEMPLE

Introductory Mathematics (1st and 2nd Years). By F. R. POTTS and G. R. CLARK. Pp. viii + 343. 11s. 6d. (Hall's Book Store Pty. Ltd., Melbourne)

This book, so the authors tell us, is written to cover the new syllabus recommended by the University of Melbourne and the Education Department of Victoria for First and Second Year Mathematics. One gains the impression that the layout and arrangement of topics is a bold innovation in the context of Australian conservatism, but, judged by English standards, the book appears as a somewhat uneasy compromise between old and new.

Arithmetic, Algebra, and Geometry are treated separately, in that order, but bound in a single volume. The reviewer feels there is a great deal to be said for this, and much prefers it to the popular arrangement of a chapter of each dotted about without organic connection. The quality of the treatment also increases in the same order: the Geometry is by far the best, and very interestingly handled; while the Arithmetic is definitely weak. The type is unduly small for an elementary book, and the binding does not look as though it would stand up to rough school use. This may, of course, accelerate the demand for a second edition, in which case it is hoped that some of the book's

weaknesses will be remedied, for it has many pleasant features. Much use is made in exercises (and this is a very good point in its favour) of a map of Melbourne printed on p. 238, but the map is so small that the street names are at the limit of legibility without a magnifying glass. This should certainly be put right: in view of the constant reference to the map, it would be better to print it on a separate folder and insert it at the back of the book. The reviewer also questions the wisdom of printing the answers to exercises immediately following them, in a book designed for young children.

The following detailed criticisms are offered in a constructive spirit. The handling of the difficult subject of fractions is rather brief, and probably more easy exercises will have to be devised by the teacher; more could well be included, and not at the end of the chapter. The various meanings of the word "average" (Chapter 6) could be brought out more clearly; it might have been worth while, after the introduction of Negative Numbers (Chapter 24), to suggest that some of the exercises on average could be reworked by the deviation method. In operations with decimals the rules given are, for multiplication, put the units digit of the multiplier under the last figure of the multiplicand; for division, either make the divisor a whole number or "standard form". The check in multiplication by counting decimal places is not given. Rough estimates are encouraged. The meaning of "significant figures" is explained only by examples, and the rule for "short" decimalisation of money not at all. We are told on p. 83 that the perimeter of a circle is approximately $3\frac{1}{7}$ times the diameter, and that the fraction $3\frac{1}{7}$ is so important that it is given the special symbol π! No decimal approximation to π is given, but even at this stage it should be correctly defined. On p. 99 we are told, it is true, that 22/7 is an approximate value for π, but the displayed formula $A = \pi r^2 = \frac{22}{7} r^2$ is far more conspicuous! The appearance of percentage *before* proportion makes the treatment of the former inadequate, though it is good to see "unitary method" relegated to its proper place. "Substraction" meets us in bold type twice—on p. 147 and p. 176. The treatment of Minus Quantities (Chapter 24) is hurried and slipshod. The Geometry begins with solid figures and construction work; plane constructions are introduced as a means to an end. The only start made with formal work is the proof of the angle-sum property. The pupil is left to make discoveries in exercises; in the hands of a good teacher this part of the book seems well-suited to the job. To say a locus is a "locality" is surely only half the story: the idea of an aggregate of all possible locations, and only those locations, must somehow be brought out. And is it wise to construct triangles with two angles and a non-included side before the angle-sum property has been formally stated? The reviewer had not met either a she-oak, a skillion roof, or a perigon, but their appearance is not less welcome for that.

The many Australian features—including a contre-jour photograph of Parliament House from the *south*—and its price—make the book unsuitable for use in this country, but writers of future English textbooks might well find helpful ideas in it, and a basis for further improvement.

H. M. Cundy

Contributions to the Theory of Nonlinear Oscillations. Vol. III. Ed. S. Lefschetz. (Annals of Mathematics Studies, No. 36)

Like the two other volumes, this consists of a rather heterogeneous collection of new papers on the subject, some for second order systems, some for nth order systems. Several concern bi-periodic systems. The method of reproduction by photographing a typescript makes the work somewhat unattractive to read, especially in the case of certain papers with very heavy formulae.

M. L. Cartwright

Plane Trigonometry. By E. RICHARD HEINEMAN. 2nd Ed. Pp. xii, 167. 22s. 6d. 1956. (McGraw-Hill)

The author is Professor of Mathematics at Texas Technological College. The aim has been teachability, and the hope is expressed that the book will be especially beneficial to students with a weak mathematical background. The scope is that of the Advanced level of the G.C.E. starting from fundamentals, but the sequence of topics is not that of our four year approach.

An outstanding feature is the care shown in organising the examples. It is claimed that a balanced coverage is obtained by working examples 1, 5, 9, etc., or other series beginning with 2, 3 or 4. The first half of each set of computation problems requires no interpolation. Answers are provided to three-fourths of the problems, but no answers are given to problems numbered 4, 8, 12, etc. Two alternative plans of usage employing 30 or 45 one-hour lessons are given, each chapter being assigned a suitable number of lessons.

The production, layout of definitions and diagrams are good. The explanations and text are good and often refreshingly informal. A very good index is provided. An adult at grips with the subject for the first time, or brushing up the work, would find the book most useful. J. W. HESSELGREAVES

Mathematical Test Papers for Upper Forms. By R. BLAMIRE CLARKE. Pp. 62. 2s. 1956. (University of London Press)

The author is the Senior Mathematics Master at the Thomas Linacre Secondary Technical School, Wigan. The purpose of the book is to provide a sequence of graded test papers for G.C.E. Ordinary Level, and especially for pupils who are taking the "mixed" papers of the type which some examining bodies call "the alternative syllabus". There are four elementary papers and eighteen harder papers, each being divided into sections A and B. The four last papers have questions on calculus.

The questions cover a wide range of topics; they are well chosen and displayed. There is a very generous supply of the longer type of question in section B, where care is needed in the interpretation. It is claimed that nearly every type of question appears again in a later paper. The printing is clear and attractive, the diagrams are good. Answers are provided. The linen back looks as if it is likely to wear well. J. W. HESSELGREAVES

Mathematics, Magic and Mystery. By M. GARDNER. Pp. 176. $1.00. 1956. (Dover Publications, New York)

This is a collection of 120 tricks, depending not on sleight of hand or collaboration, but on simple mathematical principles, some of them published for the first time, and many of them quite interesting even when you know the underlying piece of mathematics. The puzzles have been well organised into groups to emphasise common principles. There is, for instance, a big section dealing with puzzles in which an object appears to vanish when one part of a diagram is moved relative to the other; the underlying principle here is that the object which disappears distributes its part amongst the remaining objects. Related to this group are the puzzles in which a figure is cut up and rearranged, seemingly losing a fragment of its area thereby. There are puzzles using Möbius bands, topological properties of curves, scales of notation, even puzzles based on the numbers of letters in number words. There is a very nice variant of the familiar "find the number first thought of" puzzle, in which the number is obtained without the victim supplying any of the results he obtains in the calculation. Another very simple device finds a selected object out of three when the objects may be rearranged at will, by repeated transpositions. There appears to be a mistake in the puzzle on page 62; the moves on the chess board should be knight's moves, not king's moves as stated. R. L. GOODSTEIN

Engineering Mathematics. By K. S. MILLER. Pp. xii, 417. 47s. 6d. 1956. (Constable)

The author, of New York University, has taken a wide title; but the engineers to whom he addresses himself are those concerned with electrical matters and communications. Chapters 6 and 7, the final chapters of the book, deal with network theory and with random functions, stochastic and ergodic processes, "noise" and related topics of great importance to the communications engineer. The earlier chapters are therefore devoted to those mathematical disciplines not usually included in American undergraduate engineering courses which are nevertheless needed for these more professional studies. Determinants and matrices give the theoretical basis for network theory; the treatment here is somewhat concise, and the wise reader will do well to keep the theory and the applications in step by steady backward and forward reference. For random functions, the essential tools are infinite integrals and the Fourier and Laplace transforms; the author has chosen to develop the Fourier transform as a generalisation of the Fourier series, the series for a finite interval becoming the Fourier integral for an infinite interval; then the Laplace transform appears as a slightly ingenious way of dodging theoretical difficulties arising from the Fourier transform. Although I think this line of approach masks the importance of the concept of functional correspondence between image and original, it has the advantage of making clear the relations between the two transforms, so often ignored in books of this type. There is also a chapter on differential equations, with particular reference to special functions defined by differential equations; the Frobenius method of solution is fully explained. There are three appendices; one contains a proof of the validity of the Fourier series and integral representation, assumed in the main text. But the other two deal with more advanced topics, Borel sets and the Riemann-Stieltjes integral, and should serve to introduce the inquisitive reader to concepts of increasing value to the up-to-date communications engineer. The main chapters all carry some references for further reading, and some exercises for the student.

Professor Miller writes clearly, and has selected his material admirably; but he has not always the knack of isolating and emphasing the key step in an argument or the key result in a field. Hence the student working alone would sometimes be in difficulties; but for class work under supervision the book should prove very helpful. T. A. A. BROADBENT

A Mathematical Tool-kit for Engineers. By H. A. WEBB and D. G. ASHWELL. Pp. vii, 70. 10s. 1957. (Longmans Green)

This little book must be assessed in relation to its title. The mathematical techniques discussed are tools to be used by engineers and not processes to be analysed for mathematical rigour. Viewed in this sense it is a worthwhile text, but there is some doubt in the reviewer's mind, whether it can be useful before the engineer has followed a relatively advanced mathematical course. After all, tools must be used properly, and that requires training. Lord Kelvin said, "There is no useful mathematical weapon that the engineer should not learn to use," but there is a limit to what can be done in seventy pages. Nevertheless, whether the function of the book is to teach or provide ready references, and the authors seem vague about this in their preface and in their treatment of some of the subject matter, it is refreshing to see practising engineers introduced to mathematical processes which many students believe are disciplines introduced only for examination purposes.

The scope of the work is very wide. The book opens with a chapter on the solution of differential equations, particular integrals of linear equations being found by trial methods. This is followed by the best chapter in the book, on

the operator D, and includes some interesting uses of the operator for finding approximate sums of certain types of series. It is a pity that its use in finding particular integrals in " resonance " cases is not discussed. There are chapters on methods of finding integrals, Fourier series, contour integration, the calculus of variations and Bessel functions. A section on double integration is disappointing and one on Lagrange's equations is woefully inadequate, but they serve to point the way. A pleasing feature is the worked examples, which have been carefully chosen to illustrate the application of the mathematical techniques to engineering problems. In addition, each chapter contains problems for solution and the answers are provided.

In spite of obvious gaps and unavoidable omissions (there is no reference, for instance, to statistical methods) the authors have produced a workmanlike little text which deserves a place on the engineer's bookshelf or, in his kit-bag. It should fire the enthusiasm of those who read it and that may be its prime function. Clearly printed and well produced, it is good value for money in these times. A. BUCKLEY

Engineering Mechanics. By S. TIMOSHENKO and D. H. YOUNG. Pp. vx + 529. $7.50. 1956. (McGraw-Hill)

The book, of which this is the fourth edition, covers slightly more ground than an English textbook for the General Certificate Advanced Level, Applied Mathematics. It is written with the insistence on fundamentals that is a feature of Timoshenko's books on other engineering topics, and it also abounds in a wealth of examples taken from engineering practice. There are excellent figures to illustrate the practical problems both in the text and in the exercises. The two parts of the book are devoted to Statics and Dynamics, respectively, in this order, and practical units are used throughout but their relation to absolute units is fully explained. The authors lay stress on the importance of working out results in symbolic form and of substituting numbers only in the final answer; this practice is followed in all the illustrative examples. The equations of motion are handled as differential equations from the start and calculus methods are used whenever helpful.

Special emphasis is laid on the reduction of complex engineering or physical problems to such an idealised state that they can be expressed algebraically or geometrically.

Some of the more advanced topics, *e.g.* three-dimensional problems in statics, virtual work, plane motion of a rigid body are discussed in more detail than the elementary ones. D'Alembert's principle is explained in such detail that the student should not fail to understand the difference between the real forces and the fictitious force. Some teachers of mathematics do not realise that the use of D'Alembert's principle is a very powerful method because it enables us to use the methods of statics such as the polygon of forces, virtual work, Bow's notation, etc., to solve dynamical problems.

The price will put this book rather beyond the purse of the English student, but school and technical college libraries should insist on buying it, and teachers will find it a very useful source of examples. H. V. LOWRY

Aperçu de la Théorie des Polygones Réguliers. Vols. II and III. By PIERRE A. L. ANSPACH. Pp. 93–192, 193–298. No price. Privately printed by the Author. (94 Rue Berckmans, St Gilles, Bruxelles)

These two Monographs form a continuation of the volume reviewed in No. 334, and their contents are of the same general character as those of the first volume. The author does, however, open this third volume with some remarks by way of apology and explanation for the more bewildering features of his symbolism: for the strange " naming " of points he pleads the paucity of letters of the alphabet; for the use of the scale of 5 he refers to a memoir

published in 1951, which many of his readers surely will not possess. The symbols 5, 6, 7, 8, thus freed, are used for $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, $\frac{2}{3}$! Our old friend $\frac{1}{2}(\sqrt{5}-1)$ thus appears in this " numération à typographie réduite " (NTR) as $5(10^5-1)$. New friends are welcome, but when they masquerade as old ones and walk side by side with the genuine articles, it becomes very confusing. Amid the plethora of results there is some interesting material; notably the geometrical constructions involving the intersections of two conics—which can be parabola and circle—for the regular heptagon. A quotation of a typical comment of the author's may perhaps be permitted. "Un entraînement ardent nous porte, dans le domaine scientifique, au delà de nos découvertes immédiates. Nous devons obéir. Les sciences nous sollicitent vivement, nous pressent. Ce n'est pas tout que d'y faire apparaître des fleurs étranges et d'y réveiller des princesses endormies." Maybe we may find the princesses in the points named JILL, NITA, ARTHA, SOPHO, MAI in the diagrams; but what is a COQ doing in this company?

H. M. CUNDY

Radio. By JOHN D. TUCKER and DONALD WILKINSON. Vol. 3. Pp. 249. 12s. 6d. 1956. (English Universities Press)

With so many books available on this subject it is difficult to produce an original treatment. This book is in the form of notes arranged to compose a course which follows on the previous two volumes, although the book could be of use in itself. A valuable feature is the inclusion of questions and specimen answers.

The subject is so vast in scope as to render selection of material a headache. The contents page suggests a good balance, but on further reading one could wish that the authors had divided the material to form two volumes, dealing with fundamentals and practice separately.

After a fair start the first chapter on aperiodic amplifiers deals with negative feedback in too scanty a fashion: this part of the subject is of such importance as to justify a full chapter to itself. As an example no mention is made of the important difference in output impedance at the cathode and anode of a valve with current feedback, as exemplified by the concertina, and the distinction between current and voltage feedback is not made. The chapter on V.H.F. amplification makes an odd start with lines, which are not fully discussed until a much later chapter, and oscillators: the limitations of conventional valves at U.H.F. could be discussed more clearly, and no mention is made of the importance of the cathode lead inductance. There are similar bald spots in the chapter on oscillators, in which the conditions for stability are scantily described, and one would again like a complete chapter on the important class of relaxation oscillators and switching devices. The chapter on transmission lines and aerials is to be commended.

There is a tendency to a " memory " treatment of many parts of the subject which may be difficult to avoid at this level. There is also some poor type-setting in the algebra, for instance on page 3, which obscures the meaning.

A book of this kind is bound to have limitations, but this book should fill a need and be valuable to the non-university categories of student for which it is intended.

A. N. HUNTER

Lectures on the Icosahedron and the Solution of Equations of the Fifth Degree By FELIX KLEIN. Translated by G. G. MORRICE. Second revised edition, reprinted. Pp. xvi + 289. $1.85. 1956. (Dover Publications)

The first edition of Klein's *Ikosaeder* appeared in 1884, and the present volume is a reprint of the edition which appeared, in English translation, in 1914 and which was reviewed in the *Gazette*, Vol. VIII, No. 116 (1915), by the editor of that time, W. J. Greenstreet.

Readers already acquainted with this classical monograph will welcome a new reprinting. Those unfamiliar with it can be assured that here is a delightful, readable book written in a simple and leisurely style with plenty to interest the reader whether his interest is in geometry, algebra or analysis. Klein is not content to present proofs in a cut-and-dried deductive form; he seeks always to explain the processes of thought which led to the discovery of those arguments. Nor does he draw upon the results of, for example, group theory, Galois theory or projective geometry without giving plenty of explanatory material for the uninitiated.

The starting point of Part I is the consideration of the rotations of the five regular solids about their centres. In each case, those rotations which bring the solid into coincidence with itself form a *group*. The distinct groups of interest which arise are the *tetrahedral, octahedral* and *icosahedral* groups. (With these is also considered the *dihedral* group, which arises from a regular polygon in a well-known way.) Taking any one of these cases we consider the sphere circumscribing the solid and, after Riemann, label its points by complex numbers. For any of the N rotations in the group the point z moves to a new point z' and it is found that $z'=(\alpha z+\beta)/(\gamma z+\delta)$, where α, β, γ, δ are complex numbers independent of z. Corresponding to this we have two pairs of homogeneous substitutions of determinant 1. In this way our group corresponds to a group of non-homogeneous substitutions of the same order N and to a group of pairs of homogeneous substitutions of order $2N$. Simple geometrical considerations lead us to three polynomials, homogeneous in variables z_1, z_2 which are, apart from a multiplicative constant, invariant under each pair of homogeneous substitutions. A suitable combination of these invariant forms is a rational function $\phi(z)$ of $z=z_1/z_2$, invariant under all the non-homogeneous substitutions, and of degree N. It is shown that any rational function of z invariant under these substitutions is a rational function of $\phi(z)$. For any value of the parameter Z, $\phi(z)=Z$ is called an *equation* of the original group and its roots give a set of (in general N) points on the sphere, any one point of which can be obtained from any other by a rotation in the group. Conversely, any such set is obtained by giving a suitable value to the parameter. If just one root of such an equation is known, the others can easily be calculated. The fundamental problem of solving the equation $\phi(z)=Z$ (and an associated "form problem") is discussed in Chap. III by methods of the theory of functions and in Chap. IV by algebraic methods (Galois theory). The Dihedral, Tetrahedral and Octahedral equations are solved by extraction of roots. The Icosahedral equation cannot be solved by this method.

Part II—the theory of quintics—opens with a historical account of the work of Tschirnhaus, Bring, Hermite, Brioschi and Kronecker. It was proved by Ruffini and Abel that the general quintic cannot be solved by repeated extraction of roots; here it is shown that such an equation can be solved if, in addition, we are allowed the further operation of finding a root of an Icosahedral equation. The object of the book is to show how naturally and essentially the Icosahedral equations are connected with the quintics. The final picture is of a theory of quintics which unifies the earlier work in the subject.

J. L. Britton

Digital Calculating Machines and their Application to Scientific and Engineering Work. By A. G. Montgomerie. Pp. 262. 30s. 1956. (Blackie and Son, Glasgow)

On re-reading this book for the purpose of the present review, it became clear that several good points which had escaped the reviewer at a first reading require mention. The author, generally speaking, devotes himself to a close study of machines which can be purchased from the business supply

houses, that is, of desk and other hand forms of calculating machine. Apart from manufacturers' own publications, there exists no other work which gives so comprehensive a survey of what can be purchased, and, in this respect, the author has done a service to potential users of machines of this type. Unfortunately, the introductory chapter of the book contains suggestions as to a notation for representing operations of desk calculating machines which is very obscure, and which may well repel some readers at the start. Fortunately, however, where the author uses these in the text, they can be readily ignored without in any way detracting from the information which is obtainable. The author gives, with each of the machines, and particularly with punched card business machines, detailed proposals for the layout of certain calculations to be carried out on these machines. The reviewer found these extremely tedious, but it is quite possible that the tyro in the field may find the information of some use.

The seventh chapter of the book is concerned exclusively with practical numerical analysis and, if read without any accompanying knowledge or guide to the field, would give a very biased approach to this subject. The level of treatment is exceedingly low, as is indicated by the fact that the extraction of a square root by means of the iteration

$$x_{n+1}=\tfrac{1}{2}\left(x_n+\frac{a}{x_n}\right)$$

is discussed in considerable numerical detail without at any point mentioning that the process is an error-squaring one. This is particularly unfortunate because the numerical example given illustrates this point rather well.

Although in the preface the author suggests that a number of special purpose scientific computing machines will be discussed, the only detailed reference is to the Beevers-McEwen Fourier Synthesizer. This machine, whilst of considerable interest to X-ray crystallographers, is unlikely to arouse enthusiasm among mathematicians in general and, for example, the differential analyser would probably have been a more suitable object for attention.

The book concludes with a section on automatic digital calculators and the art of programming. This is probably the weakest feature, but in view of the condensation which has been necessary to include it at all, the treatment is on the whole understandable.

As might be expected from the publishers, the production of the book and its illustrations are excellent. The author, too, has a readable prose style and, where he is not engaged in discussing more abstract questions of notation, the reader will be able to progress readily through the book.

A. D. Booth

Mathematics and Computers. By G. R. Stibitz and J. A. Larrivee. Pp. vi, 228. 37s. 6d. 1957. £1 16s. 6d. (McGraw Hill, New York and London)

The senior author of the present book, George Stibitz, was one of the early pioneers of the design of automatic digital computing machines, and was responsible for several of the machines which were first produced. It was therefore with pleasurable anticipation that the reviewer first took up the present work. Unfortunately, apart from a very small section dealing with an adding circuit based on a logical principle due to Stibitz, the book does not contain information which is either new or particularly well arranged from the point of view of the present-day reader.

In the eleven chapters which, together with bibliographies, constitute the book, the subjects covered are: 1. Mathematics, computers and problems. 2. Applied mathematics and solutions. 3. Kinds of problems and where

they come from. 4. History of computers. 5. Numerical analysis. 6. Digital computer components. 7. Logical design of digital computers. 8. Analog computers and simulators. 9. Computing with random numbers. 10. Computing errors. 11. Computers at work. It is not necessary to consider the subject matter in detail, except to say that the general layout of the book closely parallels that adopted in Hartree's *Calculating Instruments and Machines*, published almost a decade ago. Computers in the sense in which they are understood in the book, are not only digital but also analogue, and naturally, in a book of this size, it is quite impossible for the authors to have covered either field with any degree of adequacy.

In various places it is assumed that the reader already has prior knowledge of certain technical aspects of the computer field, and this is unfortunate because the general standard of the work is such that it is suitable only as an introduction to the subject. It seems likely that the class of reader for whom it was originally intended was the American college undergraduate, or possible graduate, whose general learning requirements tend to be of the superficial kind.

The sections of the book which deal with digital computers and their components do not give any real impression of the field as it exists at the present time, and the structure of the computer which is mentioned in the text differs considerably from those which are in use today. Probably the best chapter is that on computing with random numbers. This subject is not readily available in any existing texts of an elementary character, and the authors have put together a most readable account, which is illustrated with numerical examples which make the way of the reader easy. The final chapter on computers at work is very disappointing. Most applications have only a few lines devoted to them, so that the chapter consists more or less of a list of problems which have been attacked. The bibliography, too, in respect of this chapter, leaves much to be desired, since the references tend to be publications of a very general character which have appeared in semi-popular journals.

The book is well produced and will certainly find a place on the shelves of experts in the field, but it is rather doubtful whether its high price will make it attractive to the general run of English readers who have less expensive and more informative books available. A. D. Booth

Ebene Kinematik. By W. Blaschke and H. R. Müller. Pp. 269. DM. 26.80. 1956. (R. Oldenbourg, München)

This exposition of plane kinematics written essentially by Müller is based on lectures given by Blaschke at Hambourg. The first chapter deals with problems of plane kinematics involving one degree of freedom and includes such topics as involutes, evolutes, and special linkages. The second chapter deals with one-dimensional problems involving integration, for example the theory of Amsler's planimeter. Kinematic problems involving more than one degree of freedom are considered in Chapter III. The final chapter deals with the theory of kinematic representation first introduced by Blaschke in 1911, in which plane motions are described in terms of quaternions whose coefficients are dual numbers.

The work abounds in numerous references to the historical development of the subject, and concludes with a detailed index and bibliography. The subject matter is classical, but in this book is gathered together for the first time a large number of topics which hitherto have been scattered throughout the literature. The authors are to be congratulated on the results of their labours, and it is to be hoped that a promised second volume dealing with three-dimensional kinematics is equally successful. T. J. Willmore

Relaxation Methods in Theoretical Physics. Vol. II. By R. V. Southwell. Pp. vi, 522. 55s. 1956. (Clarendon Press, Oxford)

It is not possible to review the present volume without saying something about the two volumes that preceded it. As Professor Southwell himself mentions in the preface to this latest volume, he started exploring the potentialities of Relaxation Methods back in 1935. The first fruits of this exploration were, however, not published until 1940, when an impressive account of work done by himself and his co-workers at Oxford appeared under the title *Relaxation Methods in Engineering Science* (R.M.E.S. for brevity of reference). Even though an attempt has been made to make the present volume self-contained when combined with Vol. I, a reader new to the Relaxation approach would be well advised to take up the study of R.M.E.S. as a useful introduction to the revolutionary change of attitude to physical problems which that approach entails. This book also contains much that, for completeness, is repeated in the later work.

As perhaps is now fairly well known, the basis of the Relaxation Method is the fact that the externally applied constraints necessary to make any trial solution a real solution for a parallel problem can readily be determined. The essence of the method lies in then finding the effect of removing such constraints systematically and in successive stages to reach finally a solution in which the constraints still remaining are negligible.

In this volume the author turns his attention to biharmonic problems, *i.e.* problems governed by the biharmonic equation

$$\nabla^4 w = Z(x, y)$$

where $Z(x, y)$ is some specified function of x and y. The continuous plane x, y is again represented by a network of points which allows $\nabla^4 w$ at any one node to be represented by a finite-difference expression involving the w's at neighbouring points—a greater number of them now because of the higher order of the derivatives. The method of approach is illustrated by a profusion of practical examples, mostly concerned with flexural and extensional problems of flat plates. The special treatment required for multiply-connected regions, as exemplified by perforated plates, is presented in detail, and emphasis is placed on the difference in technique necessitated by different boundary conditions—specified displacements on the one hand or specified tractions on the other. These matters occupy most of the first half of the book.

The treatment of eigen-value problems presented in the next seventy pages follows largely that given in R.M.E.S. but is here applied to plane problems, and much attention is given to the task of defining limits of error for the solutions obtained.

Most of the remaining pages—and they are of great interest—deal with non-linear problems, as exemplified by the large flexural distortion of flat elastic plates both under static and post-buckling loads. Here the complicating factor is the membrane or middle-surface strains introduced soon after the displacement due to pure bending can no longer be regarded as small. This ordinarily difficult problem becomes relatively simple under the Relaxation technique. The principle followed is to advance to the final solution by a series of cycles in each of which the membrane stresses are kept constant while the transverse loading brought about by their action on the curvature is liquidated. On an approximate equilibrium position being reached in a particular cycle, the associated displacement enables one to compute a new set of membrane forces, and hence a new set of transverse forces to be, in turn, liquidated in the next cycle. Many devices are described which experience has shown to be valuable for accelerating the convergence of the process. The problems of a circular plate under edge thrust, a square

plate under edge shear, the high speed flow of viscous fluid past a fixed cylinder, and plastic straining in two-dimensional stress systems have been chosen in the text as illustrations of successful practical applications.

The mathematical background of a degree in Engineering Science is what the author suggests as the necessary equipment for applying the Relaxation technique, and this is no doubt a fair assessment in so far as its application to problems of a type already covered is concerned. It is equally fair to say perhaps that, for tackling new types of problems, a somewhat wider acquaintance with Applied Mathematics is a " sine-qua-non ". D. WILLIAMS

A Text-book of Cartesian Tensors. By SHANTI NARAYAN. Pp. v, 160. Rs. 6/8. 1956. (Chand, Delhi)

The author has now some ten books to his credit, helping to meet the urgent need for good cheap text-books arising from India's vast educational programme. Primarily didactic, his books are carefully trimmed to suit their main purpose; there are few frills, and fewer digressions. The bare bones are not always attractive, but they are neatly and effectively displayed.

In the present work, an introduction on dummy suffixes leads to a long chapter on Cartesian tensors, followed by applications to geometry and to the statics and dynamics of rigid and deformable bodies. The general tensor and covariant derivatives do not appear till the final chapter, which is more of an appendix than an integral part of the book. The earlier part of the book is better than the later parts; although the fundamental bases of hydrodynamics and elasticity are concisely formulated, I doubt whether anyone coming uninformed to these disciplines would derive much benefit. Either the reader should already have some knowledge of the subjects, or, perhaps better, in a second edition the author should revise Chapter V by amplification and some informal comment.

Mr. Narayan is doing his best to help to provide text-books for India, a need strongly urged by Professor Chandrasekharan in his Presidential Address to the Conference on Mathematical Education in South-East Asia held in February 1956. T. A. A. BROADBENT

Einführung in die Theorie der Differentialgleichungen im reellen Gebiet. By L. BIEBERBACH. Pp. 281. DM 29.80. 1956. (Springer, Berlin)

The number of books on differential equations is already very large. However, as the author observes, it is less true in the theory of differential equations than in any other subject that two text-books are as like as two eggs. This book certainly contains much interesting and valuable matter. There is a very complete discussion of existence theorems for solutions of $dy/dx = f(x, y)$ by means of approximating polygons. The theory is extended to m simultaneous equations of the same type, and the behaviour of the integrals as functions of the initial conditions is discussed. There is a comparatively short chapter on elementary methods of solution; Chapter 3 contains the theory of " stationary " equations in which the independent variable does not occur explicitly, and " almost stationary " equations. The next topics are the periodic solutions of Duffing's equation

$$d^2x/dt^2 + a^2 \sin x = b \sin t,$$

and the Sturm-Liouville problem according to the method of Prüfer, with the theory of the eigenfunctions, Green's function, and expansion theorem for twice-differentiable functions (in the formula for the Green's function, for λ_n^2 read λ_n^2). The last chapter contains the elements of the theory of partial differential equations of the first order. Altogether, I think that this is a very useful book. E. C. TITCHMARSH

Remarks on the Foundations of Mathematics. By L. WITTGENSTEIN. Edited by G. H. VON WRIGHT, R. RHEES and G. E. M. ANSCOMBE. Translated by G. E. M. Anscombe. Pp. 400. 37s. 6d. 1956. (Basil Blackwell, Oxford)

This volume has been prepared from manuscripts written between 1937 and 1944. The device of simultaneous publication of the German original and the English translation which was adopted for Wittgenstein's masterpiece *Tractatus Logico-Philosophicus* has again been used, the original text occupying just under 200 pages. The five parts of the book contain observations on such problems as whether counting is an experiment, the nature of inference, the perspicuity of proof, whether the mathematician is inventor or discoverer, finitism and behaviourism, whether contradiction destroys the usefulness of a calculus, on following a rule, the role of intuition in mathematics, whether a machine can calculate, existence proofs in mathematics, whether mathematics needs a foundation, experience and timeless propositions.

Several topics are discussed repeatedly, rather in the way in which Beethoven is known to have written and rewritten a phrase in the search for the perfect form of expression. Often it is uncertain whether or not an observation is tentative and provisional, but every once in a while a remark is so apt, penetrating and indicative that it shines like a beacon across the page. Here are some examples :

(p. 50, l 7, on Gödel's undecidable proposition) " But may there not be true propositions . . . not provable in Russell's system? "—" True propositions", hence propositions true in *another* system. For what does a proposition's " *being true* " mean? (p) *is true* $=p$. (That is the answer.)

(p. 58) There is no system of irrational numbers—but also no super system, no " set of irrational numbers " of higher order infinity.

(p. 67) The *application* of the calculation must take care of itself, and that is what is correct about " formalism ".

(p. 146) Everything that I say really amounts to this, that one can know a proof thoroughly and follow it step by step, and yet at the same time not *understand* what it was that was proved. (p. 147) For it is not merely that the existence-proof can leave the place of the " existent " undetermined : there need not be any question of such a place.

To anyone who knew Wittgenstein these pages are alight with his personality, a stimulating reminder of great adventures in ideas, but the reader who comes to these ideas for the first time in these remarks may feel the need for a framework in which to fit the work. As in the *Tractatus* Wittgenstein speaks most clearly to those who have already sought to travel along the same path. Much, though not all, that he has to say in the remarks was already said, sometimes more and sometimes less successfully in his lectures in Cambridge between 1930 and 1934, and yet many discoveries which he made have still to be published.

The translation is faithful and readable without loss of the original flavour. In this Miss Anscombe had the great advantage of enjoying many discussions with Wittgenstein in the last few years of his life. Perhaps " non-temporal " on page 11 (for unzeitlich) is better rendered as timeless and perhaps to " keep mum " is not the most suitable translation of *schweigen*, but these are trifles. The best evidence of Miss Anscombe's success is that the translation reads like Wittgenstein's own spoken language. R. L. GOODSTEIN

Kettenbrüche. By A. KHINTCHINE. Mathematisch-naturwissenschaftliche Bibliothek Nr. 3. Pp. vi + 96. 1956. (Teubner, Leipzig)

This is a translation into German of an introduction to the theory of continued fractions by an eminent Russian mathematician, originally published in 1935.

K

The book may be warmly recommended. The arguments are cogently and clearly presented ; the difficulty of the first two and third parts being respectively roughly that of the easier and the stiffer parts of Hardy and Wright's *Theory of Numbers*. The first two parts are an admirable introduction to the amateur or, indeed, the number-theoretician who does not specialise in continued-fraction problems. They have naturally much in common with the corresponding portions of Hardy-Wright, but I think that Khintchine distributes his emphasis better. The striking results of the third part are, of course, more special, but they are not beyond the amateur : so far as I know they have not otherwise appeared in a book.

My only serious adverse criticism is that the book contains no references to the literature and virtually no chat about the wider aspects of the theory. Thus the result about (1) is given, and we are told it is the best of its kind, but we are not told that from a slightly different point of view it is only the prelude to one of the most remarkable results in the theory of diophantine approximation (the " Markoff chain " †), which was discovered by the great Russian academician A. A. Markoff with the aid of continued fractions. Again, to take only one further example out of several, the author points out that the arithmetic mean $\frac{1}{n}(a_1 + \ldots + a_n)$ cannot tend to a limit almost always, as the geometric mean does but does not mention his own result ‡ that in a certain sense it generally behaves like $\log n/\log 2$. However, there is a wealth of references up to 1936 in Koksma's *Diophantische Approximationen*, Perron's *Die Lehre von den Kettenbrüchen*, of which a revised edition is appearing, also contains references to the literature : it gives a fuller account of the subject matter of Khintchine's first two parts and also treats of the applications of continued fractions to other branches of mathematics which are not even mentioned by Khintchine.

The translation is on good paper—better than the original—and is well bound in boards instead of being paper-backed. § J. W. S. CASSELS

Trigonometric Series. By R. L. JEFFERY. Canadian Mathematical Congress Lecture Series No. 2. Pp. 39. 20s. 1956. (Toronto University Press. London : Oxford University Press)

This little book is of a type that is all too rare, an exposition of a field of current research in simple non-technical terms at a moment when an outstanding problem in the field has successfully been cleared up. It presents the author's Presidential Address to the Royal Society of Canada, Section III, in 1953. The lecture is in two parts, the first of which describes results which are proved in the second part.

The principal question discussed is the following : Given a function representable by a trigonometric series

$$\frac{1}{2}c_0 + \sum_{1}^{\infty} (c_k \cos kx + d_k \sin kx),$$

but which is not Lebesgue integrable, how are the coefficients c_k, d_k deter-

† Strictly speaking, Markoff's result relates to quadratic forms, but the theorem for diophantine approximation is closely related and is known by his name. There is another Markoff chain beloved of statisticians, but that is quite a different story.

‡ *Compositio Math.*, *1* (1935), 361–382. It might be mentioned that ergodic theory puts the third part of the book in a wider setting and provides simple proofs of some, but apparently not all, of the results : see C. Ryll-Narzewski, *Studia Math. 12* (1951), 74–79.

§ But, then, the original is incredibly cheap by English publishing standards. Ordered through a Cambridge bookseller my copy cost 2/-.

mined? Jeffery reviews the solutions to this problem which have been obtained in the past twenty years by Denjoy, Zygmund, Burkhill and, recently, by the Canadian mathematician R. D. James. R. L. Goodstein

Numerical Integration of Differential Equations. By A. A. Bennett, W. E. Milne and H. Bateman. Pp. 108. $1.35. 1956. (Dover Publications, New York)

Of the four chapters of this report the first two are historical and bibliographical and constitute a very valuable and detailed work of reference in a very small compass. The third chapter considers the approximate solution of the differential equations $dy/dx = u(x, y)$,

$$dy_i/dx = u_i(x_1, y_1, y_2, \ldots, y_m), \quad i = 1, 2, \ldots, m,$$

and

$$d^2y_i/dx^2 = u_i(x_1, y_1, \ldots, y_m), \quad i = 1, 2, \ldots, m,$$

by proceeding step-by-step from one end of an interval to the other, each step involving the integration of the equation over a small sub-interval. The final brief chapter records methods of solving differential equations by transition from the solution of difference equations, Ritz's method and the method of least squares in which a solution of a differential equation $I = 0$ is sought by minimising the integral $\int I^2\,dx = 0$ (a process which has led to an extension to the Legendre series of Féjer's Theorem on the mean convergence of Fourier series). R. L. Goodstein

Elements of Algebra. By H. Levi. 2nd edition. Pp. 160. $3.25. 1956. (Chelsea, New York)

The first edition of this book was reviewed by T. A. A. Broadbent in *Gazette*, XL, 67.

Levi presents a very careful account of the constructive definitions of natural numbers, integers, rationals and reals. The basis of the structure is, however, a naïve theory of sets given without any hint that the system is known to be inconsistent. Amongst minor omissions the following are to be noted. The cardinal number of a finite set is defined (p. 19) to be the name of the standard set with which a (1, 1) correspondence is established; this confuses number with nomenclature. It is not shown that a set has a unique cardinal (for two standard sets *may* be (1, 1) related until the contrary is established) and for lack of this demonstration subsequent proofs are inconclusive, *e.g.* the proof of the commutativity of addition. It is not shown that we can always find sets of given cardinals without common members; the standard sets all have common members. Finally, does not the proof (p. 30) that $26 + 3 = 29$ involve, not merely the analysis of 3 into units, but also the analysis of 26 into $2 \times 10 + 6$? And would not a proof of $29 + 2 = 31$ be more illuminating? R. L. Goodstein

Topics in Number Theory. By W. J. Le Veque. Vol. 1, Pp. 198, 44s. 1956. Vol. 2, Pp. 270, 52s. 1956. (Addison-Wesley, U.S.A.)

Volume I of this book is intended to serve as a textbook for an elementary course on number theory and to provide the preliminary groundwork required for the subjects treated in Volume II.

In the introductory chapter, a table of values of the divisor function is used to suggest problems typical of those occurring in the subject. The second and third chapters deal with the Euclidean algorithm, unique factorisation, linear Diophantine equations and elementary properties of congruences. In Chapter 4 primitive roots are introduced. Euler's theorem is strengthened by defining the *universal exponent* $\lambda(m)$ of m in terms of the ϕ-functions of the prime-power factors of m. It has the property that $\lambda(m)$ is the smallest positive value of x such that

$$a^x \equiv 1 \pmod{m}$$

for every a prime to m. A number whose order is exactly $\lambda(m)$ modulo m is called a primitive λ-root of m; there are $\phi(\lambda(m))$ of them. The chapter concludes with an application to Fermat's Last Theorem. (At end of chapter, read Section 4-4 for 4-1.)

Chapter 5 deals with quadratic residues. The exact number of solutions of the congruence $x^2 \equiv a \pmod{m}$ is obtained for any m. The law of quadratic reciprocity is proved and applied to prove a theorem of E. Trost which states that a fixed integer is a quadratic residue of every prime if and only if it is a square; here Dirichlet's theorem on primes in arithmetic progressions is assumed.

In Chapter 6 multiplicative functions are discussed and various results concerning their magnitude are obtained. The usual estimates of Chebyshev type for sums involving primes are given. More space is given to the sieve of Eratosthenes than in most works on number theory. The chapter concludes with a demonstration that, for all sufficiently large primes p, there is a quadratic non-residue of p between 1 and $\sqrt{p}$.

Chapter 7 deals with representations of a number as a sum of two, three or four squares. (At the end of the chapter it is stated erroneously that every integer can be represented as a sum of three squares, although this has been shown to be false in § 7-5.)

In Chapter 8 a full treatment of Pell's equation is given with applications to Diophantine approximation and Hurwitz's theorem. The proof of Theorem 8-11 needs some modification for example, by assuming that

$$M(\zeta) > \frac{\sqrt{d}}{(1-\delta)k},$$

since it has not been shown that $M(\zeta)$ is a member of the set of which it is an upper limit.

In the final chapter, rational approximations to real numbers are considered in greater detail by means of Farey series and continued fractions. Here the treatment is novel in that good rational approximations are defined first and are then shown to be realised by continued fractions, which arise naturally from the requirements of the problem.

The book provides an excellent grounding in number theory and is written in a stimulating manner. There are, of course, several good textbooks available on the elementary parts of the subject, and the publication of a new one might perhaps not have been justified if the book had been limited to Volume I. However, as the author states, Volume I was written to provide exactly the preliminary material needed in the more substantial and important Volume II, which no algebraic or analytic number-theorist can afford to be without.

In Chapter 1 of Volume II the reduction of binary quadratic forms is investigated, this being done by means of the modular region for both definite and indefinite forms. Chapter 2, on algebraic numbers, begins with a proof of the fundamental theorem of algebra based only on the fact that a real continuous function of two real variables attains its infimum in a closed domain. Algebraic integers, units and ideals are introduced; it is proved that the group of units has a finite basis, and an upper bound for the number of basis elements of infinite order is given.

Chapter 3 contains various applications to rational number theory such as Fermat's Last Theorem. Kummer's theorem on regular primes is proved on the assumption of Kummer's lemma regarding units of the cyclotomic field. The equation $x^2 + 2 = y^3$, and other cubic equations are also considered. (On p. 107, middle, read (18) for (19).)

The most important chapter in the book is Chapter 4, which deals with the Thue-Siegel-Roth theorem. This is proved in the following form:

Let K be an algebraic number field of degree N, and let α be algebraic of degree $n \geqslant 2$ over K. Then, for each $k > 2$, the inequality

$$|\alpha - \zeta| < \{H(\zeta)\}^{-k}$$

has only finitely many solutions ζ in K.

Here $H(\zeta)$ is the *height* of ζ, *i.e.* the maximum of the absolute values of the coefficients of the irreducible equation defining ζ, these coefficients being relatively prime rational integers. This result is a generalisation of K. F. Roth's recent improvement of the Thue-Siegel theorem (*Mathematika*, 2 (1955), 1–20, 168); Roth gave a detailed proof only for rational ζ. The proof is distinctly complicated, but is presented in a readable manner. In a review it is not possible to say briefly anything useful about the methods employed. They corroborate the reviewer's impression that, in this field, the chief difficulties—and they are considerable—arise in finding the correct ideas, and not in the delicacy of the approximations involved, which are all-important in other branches of number theory. In fact, most of the difficulties met in following the proof arise from one's reluctance to approximate with the same abandon as the author.

In places the argument is obscure (to the reviewer, at any rate), as the following remarks show.

In Theorem 4-10 (p. 136) it should be stated that N is the degree of K as this is the first place in which N occurs.

On p. 138 the deduction from $\lceil UV \rceil < M$ that $\lceil U \rceil < M$ and $\lceil V \rceil < M$ does not appear to be valid. For example, if K is the field generated by $\sqrt{2}$ and UV is a monomial with coefficient 1, it is possible that the coefficients of U and V might be $\sqrt{2}+1$ and $\sqrt{2}-1$ respectively. If m is written for the upper bound of $\lceil GW \rceil$ obtained on p. 134, this difficulty can be surmounted by observing that $\lceil \beta \rceil \leqslant m$ and that the absolute value of the smallest conjugate of β does not exceed m^{1-N}, since β is an algebraic integer, whose norm is therefore a non-zero rational integer. This gives

$$\lceil U \rceil \leqslant \lceil F \rceil m^{N-1} \leqslant m^{N+1},$$

and similarly for $\lceil V \rceil$. Thus M must be replaced by $M^{(n+1)/n}$. This entails various modifications later on. In particular, η must be replaced by something depending on N, and the upper bound of δ must be reduced. However, since the essential point appears to be that $\eta \to 0$ as $\delta \to 0$, and not the exact form of η, the proof of the theorem goes through as before.

At the bottom of p. 137 and top of p. 138 the theorem referred to should be 4-8.

Other minor points are that the last inequality of (50) requires more than $m \log(b_1+1) < \delta^2 \log q_1$ (as one might assume); on the other hand, the factor $1+\delta$ multiplying m at the top of p. 152 appears to be superfluous since it is gratuitously introduced in the preceding inequality.

In Chapter 5 the question of transcendence is considered, Mahler's classification being introduced. This is applied to π and the exponential function. In particular a transcendence measure for e^{ϑ} is obtained, where ϑ is algebraic. A general theorem of Schneider is proved and applied to prove the Gelfond-Schneider theorem.

Chapter 6 is devoted to Dirichlet's theorem and the final chapter to the Prime Number Theorem. A complex function-theoretic proof is given, which is on classical lines except that $\pi(x)$ is studied directly rather than Cheby-

shev's functions $\psi(x)$ and $\vartheta(x)$. The result is extended to primes in arithmetical progressions, it being proved that

$$\pi(x\,;\; k, l) \sim \frac{1}{\phi(k)} \frac{x}{\log x},$$

for large x and fixed k, l. The book concludes by estimating the number of integers less than x which are expressible as a sum of two squares.

There are numerous problems and exercises on the more elementary parts of the subject, and the book is beautifully printed. It forms a valuable addition to the existing works on number theory. R. A. RANKIN

Colloque sur les questions de réalité en géométrie (Centre belge de recherches mathématiques). Pp. 190. 250 fr. 1956. (Thone, Liège)

This report of the colloquium held in Liège during May 1955 opens with two introductory essays—one of them a reprint—by Montel on the almost unknown work of the Danish geometer C. Juel. To him is due the discovery that various properties of algebraic curves and surfaces depend not on their algebricity but on their order: the order of a plane curve or a surface in ordinary space being defined as the maximum number of its real intersections with a line. Juel's remarkable results were, however, obtained on unnecessarily restrictive hypotheses; and an article by A. Marchaud in the present volume shows that many of the properties hold on the mere supposition that the manifolds in question are continuous. Further results, of a more abstract character, are described by O. Haupt.

Next follow accounts by P. Vincensini and W. Fenchel of certain aspects of the theory of convex bodies. After this come articles on Harnack's problem of circuits by L. Brusotti and V. Galafassi; the former, which contains a valuable bibliography, deals with curves, while the latter is mainly concerned with Comessatti's work on surfaces. B. Segre contributes an exposition, an extended form of which can be found in his more recent publications, of correspondences between topological varieties. The book ends with a brief account by L. Santaló of density problems in integral geometry. L. ROTH

Algebraic Varieties. By M. BALDASSARI. Ergebnisse der Mathematik und ihrer Grenzgebiete, Neue Folge, Heft 12. Pp. ix, 195. DM. 36. 1956. (Springer, Berlin)

Algebraic geometers will all be heavily indebted to Professor Baldassari for the fantastic labours and the passionate devotion to his subject which can alone have rendered possible the production of this exciting book. They are the more indebted to him because of the astonishing speed with which the book was produced, which has enormously increased its value to them.

The new developments of which account has been taken can only be hinted at here. There is the striking work on foundations, initiated by van der Waerden in the 1930's, and carried on by many authors since. This alone would fill a treatise, and in fact there is a highly condensed one by P. Samuel in this Ergebnisse series. There is the work of Zariski on valuation theory, with its applications, among other things, to birational correspondences and the reduction of singularities, which merges into the work on foundations. There is Weil's work on the algebraic theory of Abelian varieties, and the recent developments of both the classical and the new abstract theory of these varieties which has heralded the rise of a remarkable Japanese school. Hodge's work on Harmonic Integrals has been given some powerful new applications, notably by Kodaira. Todd's work on canonical systems has recently been developed with intriguing new algebro-geometric methods by Segre: it is also tied up with developments in the theory of complex manifolds (the Chern

classes). On the topological and transcendental side the introduction of the new stack-theoretical ideas of Leray and the Cartan school has led to a major revolution—and another tract in this series by F. Hirzebruch. All this work is dealt with in Baldassari's book. There is some explanation of what has been done, with an invaluable list of references: there is useful and helpful discussion of the interrelations of the various topics and their position against the classical background. The author is rightly proud of the achievements of the Italian school, whose results have sign-posted the routes now being followed.

Of course, this book is not a final and definitive text: it is intended as a handbook for the research worker and a guide to the recent literature. It is for consultation rather than continuous reading. For all its inevitable imperfections, of one thing we can be certain: it would have been beyond human industry and capacity to have produced a better book so quickly, and the speed of publication is especially valuable to those struggling to keep abreast of the present torrent of important work. D. B. SCOTT

Abelsche Funktionen und algebraische Geometrie. By F. CONFORTO. Die Grundlehren der Mathematischen Wissenchaften, Bd. 84. Aus dem Nachlass bearbeitet und herausgegeben von W. GRÖBNER, A. ANDREOTTI, M. ROSATI; übersetzt aus dem Italienischen von W. GRÖBNER. Pp. xi, 276. 1956. Unbound, DM. 38.60, bound, DM. 41.80. (Springer, Berlin)

This posthumous work is a revised version of an earlier book by Conforto published in Italy in 1942. The reviewer has not seen the original version, but from the account of it in Mathematical Reviews the two editions would seem to have much the same scope. From the preface to the volume under review it would appear that, while the general plan of the work is unchanged, the revisions have been extensive, especially in the later part of the book.

The object of the book is to give an account of the theory of abelian functions and abelian varieties. The treatment is, quite deliberately, in the classical tradition, and there is no account of the modern abstract theory due to Weil. More surprisingly, there is practically no reference to the topological theory, and to the important work of Lefschetz. When this has been said, it should be added that, within its self-imposed limitations, the treatment of the subject is excellent. The exposition is conspicuous for its clarity, and for the careful way in which the motivation of the various steps in the argument is explained.

The book is very well printed, and seems to be almost free from misprints It is a valuable addition to the well-known "yellow-back" series, and may be warmly commended to geometers who wish for an insight into this important and fascinating theory. J. A. TODD

The Theory of Groups. By A. G. KUROSH. Translated from the Russian and edited by K. A. HIRSCH. 2 vols. Pp. 272, 308. \$4.95, \$4.95. 1955, 1956. (Chelsea Publ. Co., New York)

Gruppentheorie. VON WILHELM SPECHT. (Grundlehren d. math. Wiss. Bd. 82.) Pp. vii + 457. DM. 69.60. 1956. (Springer-Verlag, Berlin)

Professor Kuroš had completed the manuscript of his *Teoriya Grupp* in 1940—and a typed copy of it was available at the University of Moscow—but it was published only in 1944. Though it became quickly effective and famous in the U.S.S.R. and earned its author the Chebychev Prize in 1946, the 3,000 copies that had been printed were soon exhausted. It went out of print before the western mathematicians had become aware of its existence, and a long time had to elapse before it became accessible in this country.

In 1953 a second Russian edition appeared, greatly changed on account of the advances in the theory that the first edition had heralded and largely initiated, and also a German translation of the first edition, brought up to date by editorial notes and comments, an extended bibliography, and an appendix on some recent developments. In 1955 the German translation was reprinted and an Hungarian translation of the second edition was published, containing also the appendix of the German translation in an even more up-to-date version; and finally in 1955 and 1956 the long-awaited English translation of the second edition was published.

There have long been many books on finite groups. Kuroš writes on groups in general, and though he does not exclude finite groups and the important results peculiar to them, he treats them only as special cases and examples. The topics treated, after an extensive elementary introduction to the subject, are abelian groups, free groups, free products, direct products, group extensions, and soluble and nilpotent groups. Some topics, such as subgroup lattices, some sections on abelian groups, and an axiomatic study of the group postulates, included in the first edition had to be jettisoned in the second to make room for much new material. The whole of abstract group theory could not be covered in a book of manageable size, and a selection of material had to be made; naturally Kuroš has given preference to those parts of group theory to which he and his school have made important contributions: but these contributions have been over so wide and varied a range that the contents of the book, together with the copious references, well represent all the main directions of development in pure group theory. Though much of the material is advanced, and deep methods are used, the reader is introduced to the theory by gentle stages. Everything is patiently explained, and in words rather than in formulas: the text always heavily outweighs the mathematical symbols. Unsolved problems are stated and discussed; but an attractive feature of the first edition, namely a survey of problems and an appraisal of important lines of future advance, has been omitted in the second edition, largely because it had been so successful. Much work was done in the directions indicated by Kuroš and many of his problems were solved. Others remain as a challenge from the time of Burnside, and new ones arise daily from the new advances.

Some of the recent methods and results have not found their way into Kuroš's book yet; thus, for example, the subgroup theorem for free products due to Kuroš himself, can now be proved more simply and lucidly and so as to yield some explicit numerical formulas due to Takahasi; the proof of Gruško's important theorem is long and unnecessarily involved. However, Dr. Hirsch, the translator and editor, has provided numerous explanatory comments and added references to the most recent literature. An author index, a detailed subject index, and a very extensive bibliography, brought up to date by Dr. Hirsch and even stretched to include some papers that had not yet appeared when the second volume came out, make the book an extremely useful reference work. The editor has also added, on p. 17 of Volume I, a chronological list of the most important contributors to group theory who are no longer alive. (Otto Schreier died indeed much too young, but three years later than the list allows him.)

The translator was faced with the difficulty of not only turning idiomatic Russian into idiomatic English, or rather American, but also turning a Russian exposition into an American text-book. He did this by a number of editorial changes, rearranging some proofs, adding some notes and explanatory remarks. The editorial additions are not always marked as such, nor quite covered by a statement in the editor's preface. One could have wished that the editor had acknowledged his sources of information and the help he had from several younger colleagues as painstakingly as he has thanked

official agencies, institutions and colleagues for making his stay in the United States a pleasant one.

The printing is good and misprints are not numerous, except in the additional bibliography (Volume II). Many formulas have been taken by a photographic process from the Russian original, giving a slightly uneven effect; this was done to save cost, and it is remarkable that the price of the translation is all the same more than six times that of the original. But even at this price the book is indispensable to serious students of modern group theory.

The aims of Professor Specht's book are different. It is not a text-book, nor a reference book. The author's stated aim is to " make a selection from the gigantic abundance of results that will let the reader recognize the beauty of the theory and its manifold methods ". In this he fails: no reader will recognise the beauty of group theory in this book unless he is already well familiar with it. The style is extremely condensed, and full use is made of symbols and formulas. On many pages the formulas outweigh the text, and the demands made on the reader are invariably very high: he has to extract significance from pages after pages of intricate formal arguments, without any guidance as to the motivation and organisation of the theory. The emphasis is so even throughout that it is difficult to separate the important from the unimportant. Kuroš's book has grown out of lectures delivered in a live and vigorous school of group theory at Moscow University, and tested before a rather different student audience at an American university; Specht's book has crystallised from a lone, patient and thorough study of the modern literature on the subject. This is not to say that Specht's book lacks originality; the author has devised a novel and systematic notational scheme, which has brought in its train a generalisation and unification of many diverse results, and has thus thrown new light on some aspects of the theory.

There is naturally much overlap with the material treated in Kuroš's book. Specht gives one of the more modern, simpler proofs of Kuroš's own subgroup theorem for free products, and includes Takahasi's formula; on the other hand, the extension theory follows the lines of Reinhold Baer's investigations, whereas the now more rewarding cohomology approach—included in Kuroš's book—is neglected ("... could not be taken into account any more ", but a few references from 1946 to 1949 are listed). The proof of Gruško's theorem occupies half the space given by Kuroš to the older proof.

It is surprising to find in an otherwise so formal and systematic treatise a somewhat rudimentary system of references. There is an author index and a subject index, but no systematic bibliography. There are scant cross-references in the text, and no references to the literature at all: these are placed in a short appendix, together with a few remarks and hints for further reading. Many of the theorems are credited to their originators; but quite a few are not. Thus the theorem of Magnus confirming Hopf's conjecture for free groups (p. 371) and Higman's elegant example of a group disproving Hopf's conjecture for groups in general (p. 219) are left anonymous. Theorem 2.2.24 (p. 217) is credited to H. W. Kuhn, and prefaced by a remark that no general subgroup theorem for free products with amalgamations has been found yet: but in fact such a general theorem is among the most important results of the papers by Hanna Neumann to which the author refers, in this context, in his appendix; and the above Theorem 2.2.24 is only a very special case of this general theorem. Kuhn can claim credit for adding a Takahasi-type formula to this special case, which is, however, not given in the book.

The book is superbly printed: nothing less could be expected from the Springer-Verlag. There are letters with superscripts with subscripts with sub-subscripts (for example, p. 292), and not one of them out of place. Misprints are few: it looks as if the first three lines on p. 80 and the next four ought to

be interchanged; in the statement of Theorem 2.1.17 the second suffix k should read $k+1$. The price of the book exceeds by about 50% that of the two American volumes.

To sum up: in spite of the criticisms levelled against them, both books are to be commended and recommended as important additions to the group-theoretical literature. Kuroš is an excellent introduction to the subject, a readable and attractive exposition of it, and a very useful book of reference, and can be recommended to student and expert alike. Specht is for the initiated only, but will repay with interest the hard work to be invested in its study.

B. H. NEUMANN

Fundamental Concepts of Higher Algebra. By A. A. ALBERT. Pp. 156. 1956. (University of Chicago Press)

This little book presents a remarkably clear and concise account of the theory of algebraic extensions of a field, including the version of Galois theory evolved by Artin, and a new exposition of the foundations of finite field theory making extended use of Galois and group theory. The classical problem of establishing the existence of the roots of unity in a given finite field is obviated by the introduction of the theory of *splitting fields*; a field K is said to split a polynomial $\phi(x)$ in an indeterminate x over a field F if ϕ may be resolved into its linear factors in K, any two splitting fields being isomorphic.

The treatment of logical questions in the foundations of algebra is much more satisfactory than in most texts on modern algebra. There is, for instance, none of the usual vagueness about indeterminates. An element x of a ring A is said to be scalar with respect to a subset B of A if $bx=xb$ for every element b of B; x is said to be *algebraic* over B if there exists elements $b_0, b_1, \ldots, b_r$ not all zero, and in B, such that $b_0+b_1x+\ldots+b_rx^r=0$. An indeterminate over B is a scalar which is *not* algebraic over B. Thus for instance the real number e is an indeterminate over the ring of integers as a subring of the ring of real numbers.

The axioms for a group are taken to be:

1. Multiplication is associative.
2. For all a, b of the group the equations
$$ax=b, \quad ya=b$$
are soluble in the group.

These axioms are slightly weaker than in some well-known accounts of group theory (since the uniqueness of the solutions in (2) are not postulated) and the symmetry of the assertion that each question $ax=b$, $ya=b$ has a solution is preferable to the alternative postulation of a one-sided unit and inverse. Albert's proof of the existence of a unique identity element is, however, lacunary. The proof given runs as follows. For every a (of the group G) there exist elements e, f, such that

$$ea=af=a.$$

Then $(ea)b=ab=e(ab)$. If g is any element of G there exists b so that $g=ab$; then $eg=g$ for every g (and e does *not* depend upon g, only upon the arbitrarily chosen a). Hence $ef=f=e$ and so $eg=ge=g$ for every g.

The conclusion does not follow since the f in question is a solution of the equation $ef=e$ and *this* f is not necessarily a solution of the equation $gf=g$. The proof may be completed as follows. Let g be any member of G, and let b satisfy $be=g$, then $b(ef)=be=(be)f$, and so $g=gf$ for *any* g.

R. L. GOODSTEIN

Combinatorial Topology. Vol. 1. By P. S. Aleksandrov. Pp. xvi + 225. $4.95. 1956. (Graylock Press, Rochester, N.Y.)

Alexandroff and Hopf's *Topologie*, published in 1935, remains the classical exposition of the geometrical or "combinatorial" topology, which lies midway between the general theory of topological spaces, and modern homology theories in which only a trained eye can detect the vestiges of geometrical notions. Though no longer in the height of fashion, geometrical topology continues to flourish (see, for example, the report of the Madison Summer Colloquium on the subject in 1955). The issue in English of Alexandroff's revised version of "Alexandroff and Hopf" is therefore welcome, since there is no equally thorough treatment which keeps geometrical ideas so constantly and clearly in view. Although written in 1941 (published 1947) the book is still an excellent introduction to its chosen subject. This is particularly true of the volume under review, which contains the combinatorial but non-algebraic preparations for geometrical homology theory.

After a first chapter on topological spaces, which inevitably differs little from chapters to be found elsewhere, there follow two introductory chapters intended to serve as preliminary samples of topological reasoning. For this they are not very suitable, since both will be found heavy going by most readers. The first (Chapter II) contains the Erhard Schmidt proof of the plane Jordan's theorem, a proof, published fourteen year after Brouwer's, which enjoys an esteem not well understood by the reviewer. It here occupies 26 pages, nearly twice the space needed to develop *ab initio* the more transparent algebraic arguments. Chapter III is on the classification of surfaces. The care bestowed on every detail of a rigorous theory of "cutting and sticking", with fully worked-out examples, makes it excellent workshop practice for the later chapters; but 50 pages is rather much, especially as the method does not (like Brahana's) lead to simple computing rules for a polygon with arbitrary identifications.

With Chapter IV, on complexes, the main theme is taken up. Here will be found an ample and leisurely discussion of the geometrical groundwork of polyhedral topology, including a full discussion of simplicial subdivision. The properties of convex cells required for the theory of general polyhedral complexes are established in an Appendix.

In the final chapters this seemingly elementary apparatus is shown to be adequate, first for a proof of Sperner's Lemma, and hence of the Lebesgue Pavement Theorem, and the fixed point theorem for a simplex; secondly for quite an extensive part of the theory of dimension. With the Lebesgue definition, in terms of the order of open coverings, as basis, there are established, among others, the Alexandroff theorems on ϵ-maps and ϵ-displacements of an n-dimensional space into an n-dimensional polyhedron; the embedding theorem; the sum-theorem; and the relation of the Lebesgue to the Menger-Urysohn recursive definition.

This volume, which (after the preliminary chapters) has a lighter and easier style than "Alexandroff and Hopf", can be warmly recommended to a student wishing to make himself thoroughly acquainted with the geometrical concepts on which topology is based.

The translation is excellent, but the reader should notice that on p. 42 and elsewhere "circle" means disc and "circumference" means circle.

M. H. A. Newman

Information Theory. Third London Symposium. Edited by Colin Cherry. Pp. xii, 401. 70s. 1956. (Butterworth, London)

The volume includes in shortened form all the papers and discussions of the 1955 London Symposium on Information Theory. The 35 papers range over

most of the fields of research to which the concepts of the mathematical theory of information are now being applied, and they include the foundations of the theory itself.

It must be emphasised that these papers are contributions offered by research workers for discussion by the members of the Symposium. The papers are not polished presentations of completed work intended for laymen, but tentative and condensed formulations of new approaches to difficult problems, and they invite criticism. If the mathematical reader avoids being ensnared by the dubious speculations on meaning and knowledge, he will find in this volume much stimulation and many mathematical topics worthy of pursuit.

B. C. Brookes

Die eindeutige Bestimmung allgemeiner konvexer Flächen. By A. W. Pogorelow. Heft 3 der Schriftenreihe des Forschungsinstituts für Mathematik. Pp. 79. DM. 5.50. 1956. (Akademie-Verlag, Berlin)

This is a translation into German by J. Nass of the book by Pogorelow which first appeared in Russian in 1952. The whole book is essentially concerned with a proof of the general rigidity theorem for closed convex surfaces which states: "If two closed convex surfaces in E^3 are related by a 1-1 mapping so that corresponding curves have equal lengths, then the surfaces are congruent, *i.e.* one can be transformed into the other by a motion of E^3 which does not necessarily preserve orientation."

The problem of finding conditions which imply that two closed, isometric, convex surfaces shall be congruent had already been considered by many mathematicians, including Liebmann, Minkowski, Weyl and Cohn-Vossen. In 1927 Cohn-Vossen proved that two closed, isometric surfaces, three times differentiable, with positive Gaussian curvature are necessarily congruent. The method of proof required the differentiability and curvature assumptions, but it was evident that the question of congruence could be framed without these conditions. Herglotz, in 1943, showed that the result was valid if the regularity condition was relaxed to twice differentiable and the hypothesis of positive curvature was discarded. In 1947, A. D. Alexandrow gave a similar proof when the regularity conditions were further relaxed to once differentiable together with a Lipschitz condition.

From 1940 onwards A. D. Alexandrow and his school at Leningrad have written a series of papers introducing new methods of research in differential geometry, including the so-called gluing process. A good exposition of these methods is given by Alexandrow in his book which has recently been translated from the Russian into German, *Die Innere Geometrie der Konvexer Flächen*, Akademie-Verlag, Berlin, 1954. Pogorelow makes use of these methods to prove the general form of the rigidity theorem mentioned above. The proof is long and rather complicated, and depends upon contradictions which are shown to arise from the existence of two non-congruent isometric closed convex surfaces.

Although primarily intended as a research tool for investigating properties of surfaces which lack appropriate "smoothness", these new methods together with the general rigidity theorem can be used to solve important problems in the realm of classical differential geometry, results which, hitherto, have been inaccessible to classical methods. For this reason alone the translation of these two important Russian books into a more familiar language is more than welcome, and no doubt will lead to a more intensive study and application of these new methods in differential geometry.

T. J. Willmore

Variationsrechnung und Partielle Differentialgleichungen erster Ordnung. Volume I. By C. CARATHEODORY. 2nd Edition. Pp. 171. 1956. 14 DM. (Teubner, Leipzig)

The first edition of this famous book was reviewed by Whitehead in *Gazette* XIX. The second edition has been prepared by Professor E. Holder, who has added an account of Sophus Lie's theory of continuous groups of transformations, and Cartan's theory of outer multiplication and differentiation of differential forms. R. L. G.

Introduction to Mathematical Logic. Volume 1. By ALONZO CHURCH. Pp. 376. 1956. (Princeton University Press)

This is a revised and much enlarged edition of a book in the Annals of Mathematics Studies, published in 1944, by the best-known of the American Logicians. Church was the first to prove that pure predicate logic is undecidable, and gave to the decision problem a precise significance which had till then been lacking. The present volume, however, deals only with propositional calculus and functional calculi of first and higher orders, leaving to a projected second volume such topics as Gödel's incompleteness theorem, recursive arithmetic and axiomatic set theory.

The book opens with a long introduction which discusses Frege's famous theory of sense and denotation (somewhat dogmatically) and draws careful distinctions between the ways in which words are used in sentences. The identification (p. 17) of functions which have the same range and the same values for the same arguments, taken over from Frege, promises to give rise to difficulties in volume 2, when dealing with Godel's incompleteness theorem, since Godel showed that there are functions $f(x)$ which vanish for each value of x but are not identically zero in the sense that the equation $f(x)=0$ is unprovable (in the system in question).

Though no previous knowledge of mathematical logic is assumed, this is not a book for a beginner. It is written with great scholarship and loving attention to detail, but lacks the genial pedagogic style of the famous Hilbert-Bernays *Grundlagen der Mathematik*. The beginner could, it is true, read with pleasure and profit the numerous historical and bibliographical sections which form so important and valuable a feature of the book, but for a first account of the propositional calculus he would be advised to look elsewhere.

In less than four hundred pages a survey of propositional and functional calculi must necessarily be highly selective. Church has chosen to concentrate upon a few central theories and has been obliged to content himself with brief reference in examples to such topics as Gentzen's logic of natural inference and many valued logics.

In the account of Gödel's completeness theorem, both the original form of proof which Gödel employed and the method which was introduced by Henkin are given in full; the version of Henkin's proof is not, however, as simple as one which has recently been given by Barkeley Rosser.

There are several interesting references to the non-effective character of many famous proofs in logic, for instance, Gödel's own proof of his completeness theorem, and the existence of maximal sets in Henkin's version. Church appears to have abandoned the position of his 1934 paper in the *American Journal of Mathematics* that general recursiveness does not constitute an adequate characterisation of the notion of effectiveness, or rather to have abandoned hope of finding a better characterisation which ensures that a procedure for finding something should not be called effective unless there is a predictable upper bound to the number of steps that will be required, yet it seems not unlikely that such a characterisation is to be found in the theory of ordinal recursions. R. L. GOODSTEIN

Leçons sur les principes topologiques de la theorie des fonctions analytiques. By S. Stoilow. 2nd Ed. Pp. xvi, 194. Fr. 2,400. 1956. (Gauthier-Villars, Paris)

The first edition of this book appeared in 1937. The second edition is unchanged from the first, apart from four small notes at the end which are papers published on a similar subject by the author in somewhat inaccessible journals since the first edition appeared.

The great achievement of the author has been to lay bare the topological properties of analytic functions and in particular the invention of the interior transformation. This is a mapping of one space into another which is continuous, and transforms (*a*) open sets and (*b*) compact continua into corresponding sets in the image space. It is clear that an analytic function has these properties. Conversely, the author shows that any interior transformation from one plane into another is equivalent to a topological transformation in each plane together with an analytic transformation from one plane to the other. Thus the invariance of sets of type (*a*) and (*b*) is sufficient to characterise the analytic functions among the continuous ones topologically. It is also shown that neither (*a*) nor (*b*) alone is sufficient for this purpose.

The above fundamental theorem forms the subject of Chapter 5. Naturally a good bit of introduction is necessary before its proofs can be carried through. In Chapter I we find basic topological notions leading to the theorems regarding the invariance of dimension and open sets in a topological mapping of Euclidean space. Much of the book is also taken up with Riemann surfaces and triangulable surfaces in general. The approach of the author to Riemann surfaces is concrete rather than abstract. He regards the Riemann surface in the first instance as a generalisation of the notion of the Riemann surface of an analytic function and its various power series elements. A Riemann surface thus becomes a covering surface of the Riemann sphere. While it is true that any Riemann surface can be so regarded it is more usual nowadays to take the more general definition of a Riemann surface as a 1-dimensional complex orientable analytic manifold. In this case each point on the R.S. is no longer necessarily associated with a definite point on the Riemann sphere. Such an association can, of course, be brought about by constructing meromorphic functions on the abstract Riemann surface, as was first done by Poincaré, and after that the two definitions become equivalent.

In Chapter II the author constructs an analytic function which has a given covering surface as its Riemann surface. The Riemann surface of a function here consists of all power series elements obtained from a given element by analytic continuation with the obvious topology.

In Chapters III and IV we find a topological classification of Riemann surfaces and a proof that every triangulable and orientable surface becomes a Riemann surface on introducing a suitable topology. The general closed surface is shown to be the " sphere with k handles " having genus p. Two open surfaces are shown to be topologically equivalent if there exists a suitable correspondence between their ideal boundary elements and they have the same genus, finite or infinite. An open surface can always be considered as a sphere with a finite or infinite number of handles and holes.

In Chapter V we have the fundamental theorem on interior transformations and in Chapter VI some consequences of it. It is in this last chapter that one might perhaps wish that the author had kept his readers informed of developments since the first edition. Thus on p. 125 the author states as an open problem the construction of a univalent function with a totally discontinuous perfect set of singularities. There is a footnote on how M. Denjoy thinks this might be done, but no reference to the fact that Ahlfors and Beurling did it in *Acta Mathematica* 83 (1950). There has also been important work by Cartwright and Collingwood on the general subject of this book. Only a

paper by Heins on the interior mapping of an orientable surface into the sphere is referred to.

Stoilow's book is still very frequently referred to by all those working on the borderline between topology and function theory. The book may be regarded as the basic manual in this region, and those libraries and institutions who are interested in the field but do not possess the first edition will welcome the present opportunity to acquire this book. W. K. HAYMAN

Matrix Calculus. By E. BODEWIG. Pp. 334. 26.50fl 1956. (North-Holland Publishing Company, Amsterdam)

This is not " just another " book on matrices and linear algebra. On the contrary: the emphasis lies on the word *calculus*. It is the first modern treatment of the subject, written entirely from the viewpoint of practical applicability. The three central problems connected with matrices that occur in computing practice are the solving of linear equations, the inversion of matrices, and the finding of eigenvalues. (They are, of course, not independent of each other.) To each of these problems one separate part of the book is devoted. The reader will find a wealth of information on the many methods, both direct and iterative, that have been proposed and have been successfully applied in practical work. The treatment by automatic machines, punched cards and electronic computers is fully covered, each time with an extensive discussion of the preliminary transformations that will make the problem more amenable to machine computation, and with estimates of the numbers of operations to be performed. The author gives a critical assessment of the advantages and disadvantages of the various processes.

The first part of the book contains a rapid, if somewhat condensed, survey of the underlying theory. The matrices all have their elements in the field of real or complex numbers. The operations are described rather elegantly by a systematic use of pre- and post-multiplications by suitable unit vectors and matrices. Emphasis is again laid on numerical checks of the computations to be performed and on estimates of the errors both inherent and accidental.

The book does not have much in common with the standard undergraduate University course in Linear Algebra. It is all the more valuable in an advanced course on numerical methods and will be an indispensable source of information for the " matrix engineer " who has to tell the automatic machine what it is expected to do and has to prepare its work and evaluate its results.

K. A. HIRSCH

Structure of a Group and the Structure of its Lattice of Subgroups. By M. SUZUKI. Pp. 96. DM 16.50. 1956. (Springer, Berlin)

This slim volume is part of a new series of group-theoretical monographs, edited by R. Baer, within the framework of the Ergebnisse der Mathematik. It contains a complete survey of all that is known at present about the lattice of subgroups of a given group. The history of the application of lattice theory to group theory can be said, somewhat paradoxically, to date back to 1928, long before the birth of lattice theory proper. It was then that Ada Rottländer discovered that two finite non-abelian groups with isomorphic subgroup lattices or, as it was expressed at the time, with precisely the same subgroup " situation " nevertheless need not be isomorphic. This cannot happen if one of the two groups is finite and abelian. Around this theme the main problems of this somewhat remote branch of group theory have been developed: What can be said about the structure of a group if its lattice of subgroups is subjected to certain restrictions (for example, the distributive or the modular law)? What classes of groups are determined to within iso-

morphism by their subgroup lattice? (This is the case, for example, in abelian groups of rank not less than 2, in free groups and free products, etc.)

The book gives, as mentioned above, a full report of the results obtained so far, with proofs of the major theorems. The author himself has been a successful contributor to the theory and can draw from unpublished work. But in the reviewer's opinion the subject matter is of rather specialised interest, even within the contemporary theory of groups. The deepest application of lattice theory to group theory, namely in the theory of direct decompositions (Ore-Kurosh), is not treated here, because it does not fit into the author's programme. The short hey-day of lattice theory is now over, and the theorems quoted in this volume are certainly not very exciting. This judgment is, of course, an expression of personal taste and may not be shared by others, especially those (few) who work actively in the field. K. A. Hirsch

Die Theorie der Gruppen von endlicher Ordnung. By A. Speiser. Vierte, erweiterte Auflage. Pp. 207. Fr. 26. 1956. (Birkhäuser, Basel)

The first edition of this famous book, in the "yellow" series of mathematical monographs of the Julius Springer Verlag, is 35 years old. Further editions appeared in 1927 and 1937 and a photographic reproduction in the United States in 1945. During this long period many new branches have been added to the general theory of groups, some of which now stand in the centre of the research interests. But since Speiser's book contains some valuable chapters on aspects of group theory that cannot be found in other textbooks, the transfer to the Verlag Birkhäuser is a welcome way of preserving the fine features of this classical treatment, while at the same time making room for a more modern approach. The Springer Verlag has recently added to its series a new volume on group theory, with the main emphasis on infinite groups, by W. Specht.

The present fourth edition does not differ in its main body from the third apart from corrections of a few inaccuracies. The chapters on the symmetries of plane ornaments, with many illustrations, and on the crystal classes and space groups are as attractive as they were twenty years ago. Professor Speiser has written an appendix in which he again stresses some geometric implications of group theory. He treats in detail the problem of covering a plane with regular polygons. In the elliptic plane there are the five solutions corresponding to the five Platonic regular polyhedra, each with a finite group. In the Euclidean plane the only three possible solutions are the coverings by equilateral triangles, squares, and regular hexagons, each with an infinite group. But in the hyperbolic plane there are all the solutions that arise from Klein's Grenzkreisgruppen in the theory of automorphic functions. A few of the possibilities are illustrated in the text, and a pretty frontispiece, printed in seven colours, shows a fascinating geometric representation of the simple group of order 168. K. A. Hirsch

Einfuhrung in die Verbandstheorie. By H. Hermes. Pp. 164. DM. 19.80. 1955. (Springer, Berlin)

This book presents a straightforward, well-written account of the theory of lattices without any unusual features.

The opening chapter deals with the usual introductory topics, partially ordered systems, a detailed study of homomorphisms of lattices and the notion of completeness. In the next chapter, the reader is introduced to the main types of lattice and the ideal theory of these types. The chapter closes with MacNeill's theorem on the imbedding of an arbitrary partially ordered system in a complete lattice.

The third chapter is devoted to Modular lattices. After some generalities,

a detailed account of the lattice of linear sub-spaces of a projective space is given, culminating in a lattice-theoretic characterisation of such a lattice. The second topic of the chapter is the lattice of congruence relations on an abstract algebra with a unit, this being a modular lattice if the congruences are permutable. The theory is developed as far as a form of Zassenhaus' Lemma and the related form of the Jordan-Holder Theorem. The chapter closes with a brief account of the theory of linear dependence from the lattice theoretic point of view.

The fourth chapter is a study of distributive lattices and Boolean Algebras, and their representations, while in the last chapter a number of special topics are discussed. D. REES

The Construction and Study of Certain Important Algebras. By C. CHEVALLEY. 1955. The Mathematical Society of Japan.

This little book contains the substance of a series of lectures given by Professor Chevalley in the University of Tokyo in the Spring of 1954. It is an account in extremely abstract terms of the elementary theory of certain algebras over commutative rings (the added generality in Chapter I, where the ring of scalars is not assumed to be commutative appears to the reviewer to be spurious, since the algebra axiom $\alpha(xy)=(\alpha x)y=x(\alpha y)$, where α is a scalar and x, y are elements of the algebra, implies by taking $x=\beta\,.\,1$, where 1 is the unit of the algebra, that $\alpha\beta\,.\,y=\beta\alpha\,.\,y$ for all y in the algebra, so that the ring of scalars is effectively commutative).

The first chapter is concerned with graded algebras for the most part, but commences with an account of the free " algebra " with a given set of generators over a ring A. The definition of this algebra follows the usual form of the definition of a free algebraic system, except that the term algebra is misleading since if A is not commutative it is not an algebra. The remainder of the chapter is termed with graded algebras, the author adopting a definition considerably more general than the usual one. A graded algebra E over a (commutative) ring A with additive degree group D is an algebra E which additively is the direct sum of A-modules E_d, where $d=D$, and subject to the condition that $E_{d'}\,.\,E_d=E_{d+d'}$. The end of the chapter is devoted to the study of derivations of E into another graded algebra E' with the same degree group, relative to a sub-group $D+$ of D of index 2. This is a generalisation of the notion of exterior differentiation in the graded ring of differential forms on an appropriate topological space. In the second chapter, the author introduces the idea of the tensor algebra of an A-module M, where A is a commutative ring. Here the extremely important notion of universality is introduced. The definition of the tensor algebra is effectively that the tensor algebra $T(M)$ of M over A is the unique algebra with the property that it contains M, is generated by the elements of M and such that every module homomorphism of M into an A-algebra E can be extended to a ring homomorphism of $T(M)$ into E. The author derives the theory of the tensor product of two modules by considering the tensor algebra of the direct sum of the modules, and finishes with a short account of the tensor product of two semigraded algebras (an algebra being semigraded if the degree group consists of two elements).

The last two chapters are concerned with the theory of Clifford algebras. Such an algebra is determined by a module M over A and a quadratic form $f(x)$ on M with values in A. The Clifford algebra of f over M is then defined to be the quotient algebra $T(M)/K$, where K is the two-sided ideal generated by the elements $x^2-f(x)\,.\,1$. If $f(x)=0$ identically, this yields the exterior algebra of M. The two chapters contain a number of generalisations of results familiar in the theory of differential forms, as well as brief indications of the relationship to the theory of determinants and spinors. D. REES

Vorlesungen über höhere Mathematik. Part I. By A. Duschek. 2nd Edition. Pp. 440. 81s. 6d. 1956. (Springer, Vienna)

The first edition of this book was reviewed by T. A. A. Broadbent in *Gazette* XXXIV, 145. A thorough revision has been undertaken for the new edition some chapters having been completely rewritten, the better to achieve the author's aim of a presentation which shall be as clear and intelligible as possible without the smallest sacrifice of rigour. The principal change which has been made is to bring the chapter on infinite series into Part I, and to transfer the theory of probability to Part II.

The book achieves its objectives very successfully. The treatment is thoroughly sound and trustworthy and nothing seems to have been forgotten. Every discussion is clear and intelligible, and presented very skilfully. As an example of the text's thoroughness one may mention the proof (p. 75) that $f(x)$ is continuous at x_0 if $f(x_r) \to f(x_0)$ for all sequences x_r which tend to x_0; that it suffices for the theorem for $f(x_0)$ to be just a limit point of $f(x_r)$ is not, however, stated. There are many attractive new variants of proofs of elementary theorems, for instance the proof (p. 347) that the terms of an absolutely convergent sequence may be rearranged without disturbing the sum, and the proof (p. 317) of the Descartes rule of signs for polynomials with real coefficients. By and large, however, the treatment is more noteworthy for accuracy than for elegance.

There are numerous biographical details included about the principal authors cited in the text. It is pointed out that the rule

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)},$$

usually attributed to Hermite, was in fact discovered by Bernoulli, but Cardan is given the credit for Tartaglia's formula for the roots of a cubic.

The contents of this volume include the real number system (with a proof of Dedekind's theorem, but lacking an account of the arithmetic of real numbers), bounds, limits and convergence; continuity and differentiability, mean-value theorem and Taylor's theorem; the elements of the theory of Riemann integration, properties of the elementary functions; differential geometry; roots of polynomials, finite differences and interpolation, the theory of series including Fourier series. As for the introduction of the elementary functions, the logarithmic function is defined by the integral $\int_1^x (1/t)\,dt$ and the exponential function as the inverse of the logarithmic function, but surprisingly and disappointingly the author contents himself with a geometrical definition of the circular functions. R. L. Goodstein

Automata Studies. Edited by C. E. Shannon and J. McCarthy. Annals of Mathematics Studies, Number 34. Pp. 286. 1956. (Princeton University Press; Cumberlege, London)

The essays in this collection fall into three groups. The first group deals with automatic machines having a finite number of possible internal states. This group contains (amongst others) papers by J. von Neumann and S. C. Kleene, dealing with the analysis of events in " nerve-nets " by elementary mathematical logic. A *nerve-net*, for the purpose of the analysis, is an arrangement of a finite number of neurons (nerve cells consisting of a soma from which nerve fibres lead to *end bulbs*) in which each end bulb of any neuron impinges on the soma of not more than one neuron. Each end bulb is either excitatory or inhibitory. Neurons on which no end bulb impinges are called input neurons. At equally separated moments of time each neuron of the net is either firing or quiet. Inner (not input) neurons fire at time t if a certain number h (called the threshold of the particular neuron) of the excitatory end

bulbs and none of the inhibitory end bulbs impinging on it belong to neurons which fired at time $t-1$. The following diagrams illustrate the way in which

elementary neurons may be interpreted logically. These diagrams (leading from left to right) depict respectively a threshold 2 neuron with two excitatory inputs, a threshold 1 neuron with two excitatory inputs and a threshold 1 neuron with one excitatory and one inhibitory input. The diagrams may therefore be taken to represent, in turn, logical conjunction since the first neuron fires at time t if and only if *both* inputs fire at time $t-1$; logical disjunction since the second neuron fires at time t if *either* of the inputs fires at time $t-1$; the proposition a and not-b (*i.e.* a does not imply b) since the third neuron fires at time t only if a fires and b does not at time $t-1$.

The second set of papers deals with Turing machines, which are automata with unlimited memories. The accounts of these machines in these essays are remarkably clear and afford an excellent introduction to the theory. Shannon's paper shows that a universal Turing machine, a machine which carries out the operations of any assigned machine when supplied with the code number of that machine, can be constructed with only two internal states, or alternatively with only two tape symbols. Other papers consider the effect of introducing random elements in machines.

The third set of essays is concerned with the design of automata which will simulate (in some sense) the operation of the human brain, and with the problem of concept formation. R. L. Goodstein

Problems and Techniques of Applied Mathematics. By B. Friedman. Pp. ix, 315. 64s. (John Wiley, New York; Chapman and Hall, London)

The subject of this book is the theory of operators in linear spaces and its application to the theory of the differential operators of applied mathematics. Two chapters, on Linear Spaces and on the Spectral Theory of Operators, are devoted to the abstract theory; three, on Green's Functions, Eigenvalue Problems of Ordinary Differential Equations, and Partial Differential Equations, respectively, to the applications.

The author has written his book explicitly for the applied mathematician, and his object has been to make the subject accessible to him. For this reason he has consciously written the abstract portions of the book with a standard of rigour lower than that to which one is accustomed in expositions aimed at pure mathematicians, though definitely higher than that usual in many books on quantum theory in which this subject is discussed. His method of exposition in this part of the book is semidescriptive: thus, in the chapter on spectral theory, a fairly complete account of the spectral theory in finite dimensional spaces is given, and is used as an analogy for the situation for operators in function spaces; some of the theorems for these spaces are proved in appendices, others are left unproved.

On the whole the author has carried out his aims very successfully. There are perhaps some points at which greater rigour, and more definition, would have been helpful even to, say, a physicist; for example, in the first chapter

particular cases of convergence in an infinite dimensional space are discussed without any criteria being given as to what constitutes convergence in the space. Another rather unhappy compromise is that of using the formula for the scalar product in a real Hilbert Space, while allowing the space to have complex scalars; a procedure which would, if adopted consistently, make it impossible to talk of the limit of a sequence in the space. Certainly, the author avoids the pitfalls which lie on this way of exposition; and is careful to state his formal theorems for real scalars only; but this technique does seem to risk confusing the less wary reader.

The sections on differential equations are well done, a considerable number of examples being given to illustrate the theory. The delta function is used for the construction of the Green's functions, and its use is justified by an appeal to the theory of distributions. The explicit solutions of eigenvalue problems are carried out by the method of contour integration, following Titchmarsh.

The book is well produced, and has an adequate index. It can be recommended to Applied mathematicians not interested in a rigorous account of the abstract theory and to pure mathematicians interested in the applications and prepared to supply the rigour. J. B. L. Cooper

Topological Dynamics. By W. H. Gottschalk and G. A. Hedlund. Pp. v, 151. $5.10. American Mathematical Society Colloquium Publications, Vol. XXXVI. 1956.

Few prospective readers will expect to find in this book any close connection with the studies of rolling bodies and vibrating strings which dynamics suggests to most of us; these connections, the authors explain in the preface, are historical, and the book has little mention of physical dynamics.

The most general view of a dynamical system in classical theory is that which regards it as represented by a point in its phase space; the equations of motion of the system, in the canonical form, govern the motion of the representative point in such a way that the position of the point at one instant of time determines completely its position at all later (or earlier) times. The aggregate of all possible states of a system is then represented by a cloud of points, and these points move in time rather like the particles of an incompressible fluid. This point of view was used by Poincaré, and later by Birkhoff, to discuss under what conditions dynamical systems tend to return to their initial configurations or to tend asymptotically to certain configurations.

This point of view is that a dynamical system, or rather its set of representative points in phase space, is subject to a group of transformations, each element of the group being the transformation brought about in the phase space by the development of the system for a time corresponding to that element. The work of American and Russian mathematicians has shown that many of the results which are valid for the dynamical systems do not depend on the nature of the differential equations by which the system is governed; and indeed that they can be generalised to cover sets subjected to a group of transformations irrespective of whether the transformations are generated by differential equations or of whether the group depends on only one parameter. In these generalisations it is only the similarity of the type of problem studied to those of the dynamical system which differentiate topological dynamics from other parts of the theory of topological groups. These problems are those concerned with the nature of the paths—*i.e.* the set of transforms—of a point under the group; the manner in which the space is divided by these paths, their periodicity and asymptotic behaviour.

The present work gives an account of this subject as it has developed since it parted company with physical dynamics: since, that is to say, the state

of development presented by the ninth volume in the Colloquium series, that of G. D. Birkhoff on *Dynamical Systems*. It is written in a style which is very precise and accurate, but extremely terse: one might almost say telegraphic. Definitions and theorems are given and proved in a highly abstract style, with little explanation of their meaning, save for references to the literature at the end of each chapter. Prerequisites for its reading are an acquaintance with the topology of Bourbaki and with the elements of the theory of topological groups. A bibliography and index are provided. J. B. L. COOPER

The Mathematics of Diffusion. By J. CRANK. Pp. vii, 347. 50s. 1956. (Clarendon Press)

This is a big book, and of an unusual kind. It is written by an industrial mathematician, and is solely concerned with solutions of the standard differential equation of diffusion

$$\operatorname{div}(D \operatorname{grad} C) = \partial C/\partial t,$$

where D is the diffusion coefficient and C the concentration. The reason for the importance of this equation is that it occurs in all processes where two liquids mix together, and also in such processes as the freezing of water, where the boundary moves at a rate proportional to the rate of freezing. Dr. Crank first deals with the equation in which D is constant, showing how both the method of separation of coordinates and the Laplace transform may be used to give solutions for linear, cylindrical and spherical flow. Then he allows D to vary, continuously or discontinuously, with C, and introduces moving boundaries. The analysis is everywhere exceedingly clearly set out, and in no other account that I can remember is there so firm a convention that every result obtained theoretically must immediately be shown graphically, and in such detail that a mere experimenter could make use of it, even if he didn't understand the processes by which the result was obtained. To a person not actively concerned in diffusion problems, there is a certain dullness about the book; but this is inevitable in a specialised account when the author is anxious to leave nothing out. It is also rather inevitable that in the first part of the book, where the author is describing the methods of solution of the diffusion equation, there should be a good deal of overlap with the books by Carslaw and Jaeger. But this is a very clear account which, while not being particularly suited to undergraduates, may easily prove invaluable to the growing number of mathematicians who are now entering industry, and having to tackle problems of a similar kind to those treated here. The layout and printing of this book are magnificent. When I think of what these long formulae might have looked like, I am glad that this is an Oxford book! C. A. COULSON

Théorie Générale de l'Equation de Mathieu et de quelques autres equations differentielles de la mecanique. By R. CAMPBELL. Pp. 271. 2,400 Fr. 1955. (Masson, Paris)

This book on the general subject of Mathieu Functions approaches the subject in quite a different manner from that of MacLachlan's treatise on the same subject and is a valuable addition to the literature in this field. The main aim of this book is to explore deeply the several new techniques which have been developed for the study of Mathieu Functions and to exploit these techniques in the investigation of allied differential equations.

A valuable feature of the book is its presentation at the end of each chapter of a complete set of tables summarising the results of the chapters and in this field, where the number of solutions is so large, these provide useful concise information. It is to be regretted on the other hand that the large number of errata occupying two pages at the beginning of the book could not have been incorporated in the book itself and also that this set of errata is by no means complete. T. V. DAVIES

Modern Mathematics for the Engineer. Edited by E. F. Beckenbach. Pp. 514. 56s. 6d. 1956. (McGraw-Hill)

This volume contains 19 lectures on branches of modern mathematics given in a series of lectures arranged by the departments of engineering at the University of California. The lectures are by many distinguished mathematicians, including R. Courant, S. Lefschetz, N. Wiener, D. H. Lehmer, I. S. Sokolnikoff, and R. Bellman. The emphasis in most of the articles is on numerical methods and linear approximations, but there are some remarkably good expositions of underlying theory.

Especially noteworthy for their clarity are the lectures on the Calculus of Variations by M. R. Hestenes, on the Theory of Games by H. F. Bohnenblust, on the Theory of Dynamic Programming by R. Bellman, on Matrices in Engineering by L. A. Pipes, and on Relaxation Methods by G. E. Forsythe. Lehmer's lecture on High Speed Computing Devices makes some interesting comparisons between analogue and digital computers, but is disappointingly brief on the subject of machine programming.

Many of the articles have historical sections and valuable bibliographies. At a price which is quite reasonable by present-day standards this volume provides an impressive survey of a very wide range of mathematical methods.

R. L. G.

THE MATHEMATICAL ASSOCIATION

The fundamental aim of the Mathematical Association is to promote good methods of Mathematical teaching. Intending members of the Association are requested to communicate with one of the Secretaries. The subscription to the Association is 21s. per annum and is due on January 1st. Each member receives a copy of the *Mathematical Gazette* and a copy of each new Report as it is issued.

Change of address should be notified to the Membership Secretary, Mr. M. A. Porter. If copies of the *Gazette* fail to reach a member for lack of such notification, duplicate copies can be supplied only at the published price. If change of address is the result of a change of appointment, the Membership Secretary will be glad to be informed.

Subscriptions should be paid to the Hon. Treasurer of the Mathematical Association.

The address of the Association and of the Hon. Treasurer and Secretaries is Gordon House, 29 Gordon Square, London, W.C.1.

SOUTHAMPTON AND DISTRICT BRANCH

Report for the Session 1956–1957

The beginning of this session was marred by the sudden death of our Secretary, Dr. F. G. Maunsell, who had served the Branch for a period of about twenty years. A small presentation was given to his widow in token of our appreciation of his work for the Branch and Association. This delayed initial arrangements and as a result we only had five meetings this year instead of the usual six.

Thursday, 15th November, 1956. The meeting was preceded by the Annual General Meeting of the Branch when the following were elected as members of the Committee for 1956–1957 : *President*, Dr. L. J. Stroud ; *Vice-President*, Mrs. W. S. P. Edmunds ; *Secretary and Treasurer*, Mr. J. C. F. Fair *University Representative*, Miss N. Walls, also Miss P. M. Pickford, Mrs. M. Laing and Mr. T. A. Jones. Then our President, Dr. L. J. Stroud, read a paper on " The Place of Mathematics in the Grammar School Curriculum ", in which he showed that the development of basic skills is an essential prerequisite of the desired progress towards the power of generalisation.

Thursday, 7th February, 1957. Miss E. M. Spark talked about " Some Experiments in General Mathematics for the Non-specialist ", basing her ideas on her experience with Training College students of all grades.

Tuesday, 19th March, 1957. Mr. A. R. Pargeter gave an illustrated talk on " Plaited Polyhedra ", in which he demonstrated the plaiting of neat firm solids without the use of glue! His models were eagerly examined after the meeting.

Tuesday, 14th May, 1957. Dr. D. M. A. Mercer gave a demonstration lecture on " Noise and its Measurement ". This was illustrated visually, as well as audibly, using a cathode ray oscilloscope and an octave band analyser.

Tuesday, 25th June, 1957. A report of the " Conference of Industrialists and Teachers of Mathematics " held in Oxford at Eastertime was presented. This took the form of a symposium by three members of the Branch, Mr. R. G. Taylor, Miss H. Bromby and Mr. T. A. Jones, and was followed by a lively discussion.

The Branch now consists of 32 Members and 20 Associates ; and has an average attendance of 38, which includes visitors who see the meetings advertised in the Local Education Circular.

J. C. F. Fair, *Hon. Sec.*

NOTTINGHAM AND DISTRICT BRANCH

Report for the Session 1956–1957

The Annual Meeting was held on 3rd November when short papers were presented by members after the formal business had been concluded. Dr. Power spoke on " Introducing Vectors ", Dr. Jackson on " Mathematical Difficulties of First Year University Students ", Mr. Reynolds on " Laplace and Lagrange ", and Mr. Buckley on " Introducing Mechanics ".

The Spring Meeting took the form of a One Day Conference held on 9th March at Nottingham University. The speaker at the morning session was Mr. M. J. Moroney on " Uses of Statistics in Industry ". In a lively and stimulating way he presented a picture of the demands made on Industrial Statisticians, pointed out the difference between theoretical methods and the

more experimental approach commonly used in Industry, and drew on his own wide experience for typical examples. The afternoon session was devoted to a discussion on " The Place of Formal Geometry in the School Curriculum ", which was introduced by Mr. Parr, who defended the flexibility of treatment allowed by the Alternative Syllabus. Mr. Fleming deplored the fact that pupils could now pass the G.C.E. examination knowing little formal Geometry and felt this to be a pandering to mediocrity. Geometry had a unique value in developing style and elegance accompanied by a sense of satisfaction. A general discussion followed.

At the Summer Meeting on 27th June Mr. Griffiths spoke on " Careers for Mathematicians " and described the kind of work available in Industry for the man who combined engineering knowledge with mathematical skill. He spoke of the difficulties experienced by many in the transition from the exactness of University Mathematics, and suggested some changes of emphasis in the School curriculum which would help future industrial mathematicians.

The Officers of the Branch were :

President, Dr. G. Power ; *Vice-President,* Mr. K. R. Imeson ; *Treasurer,* Mr. C. R. Swaby ; *Secretary,* Mr. F. E. Chettle.

F. E. Chettle, *Hon. Sec.*

CARDIFF BRANCH

Report for the Session 1957–1958

Officers :

President, Dr. D. G. Taylor ; *Treasurer,* Mr. R. A. Jones ; *Secretary,* Mr. W. H. Williams.

Meetings :

Monday, 14th October, 1957. Mr. R. A. Jones of the Howardian High School gave an address on " Some Elementary Ideas in Graphical Mathematics ", which included a novel approach based on similar triangles.

Monday, 11th November, 1957. Dr. H. O. Foulkes of the University College, Swansea, gave an address on " Some University Entrance Scholarship Problems in Pure Mathematics " which provoked a lively discussion.

Monday, 27th January, 1958. Dr. P. White of the University of Reading, gave an address entitled " What is the Matter with Negative Mass ", an entertaining and instructive talk.

Monday, 17th March, 1958. Mr. D. G. Kendall of Magdalen College, Oxford, gave an address : " Some Problems of Mathematical Epidemiology ", considering first the case of an epidemic in an isolated population and then generalising the problem by introducing a population density.

W. H. Williams, *Hon. Sec.*

EXETER BRANCH

Report for the Session 1957

The Annual General Meeting was held on 26th January, 1957. Professor T. Arnold Brown of Exeter University—the retiring President—addressed the Branch on the subject " Problems of Inference ".

Mr. A. P. Rollet was elected *President*. Other Officers were: *Vice-President*, Professor T. Arnold Brown; *Hon. Treasurer*, Miss L. G. Button; *Hon. Secretary*, Miss N. A. Comerford.

Other Meetings for the year were held as follows:

12th April, 1957. Miss E. O. Wolstenholme on "Anticipating VI Form Work". This Meeting was held in conjunction with the Institute of Education, University of Exeter.

17th May, 1957. Mr. J. T. Combridge of King's College, London, on "Is Mathematics for the Many or the Few?"

22nd June, 1957. Dr. H. Martyn Cundy on "The Mathematics of And/or"—an interesting introduction to binary logical units.

3rd November, 1957. Mr. R. C. Lyness on "Some Top Whole Number Problems".

There are about 50 members of this Branch. Meetings have been well attended and a considerable number of students and VI Form visitors have been welcome.

N. A. COMERFORD, *Hon. Sec.*

NEW SOUTH WALES BRANCH

REPORT FOR 1957

During 1957 four meetings of this Branch of the Mathematical Association were held.

In first term, the meeting of 3rd May discussed the Mathematics Examination papers set for the 1956 Intermediate and Leaving Certificate Examinations.

Two meetings were held in second term. At the first, on 14th June, Professor Bullen, of the Applied Mathematics Department, University of Sydney, gave an address on "The International Geophysical Year". The second meeting was held on 2nd August, when Mr. R. W. Hundt addressed the meeting on the topic "The Brighter Mathematics Lesson".

In third term, the meeting planned for 20th September took the form of a visit to the Silliac Computer, at the Physics Department, University of Sydney. The Branch would like to thank Dr. Bennett and Mr. Williams for arranging this most interesting visit.

The Annual General Meeting was held on 1st November, when Mr. I. Kershaw and Mr. B. Mudie led a discussion of the 1955 Leaving Certificate Syllabuses in Mathematics I and II (these syllabuses to be reviewed by the Mathematics Syllabus Committee in 1958).

In addition to these Branch meetings, the Executive Committee met twice: the meeting of 13th March to arrange meetings for 1957, and the meeting of 1st November to make recommendations for Office-Bearers for 1958.

A. R. BUNKER, *Hon. Sec.*

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